

AD-A047 781

AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OHIO SCH--ETC F/G 9/2  
AN ALGORITHMIC APPROACH FOR REDUCING KINEMATICALLY SIMILAR DIFF--ETC(U)  
DEC 77 K DOLAN

UNCLASSIFIED

AFIT/GCS/MA/77D-1

NL

1 OF 1  
AD  
A047781

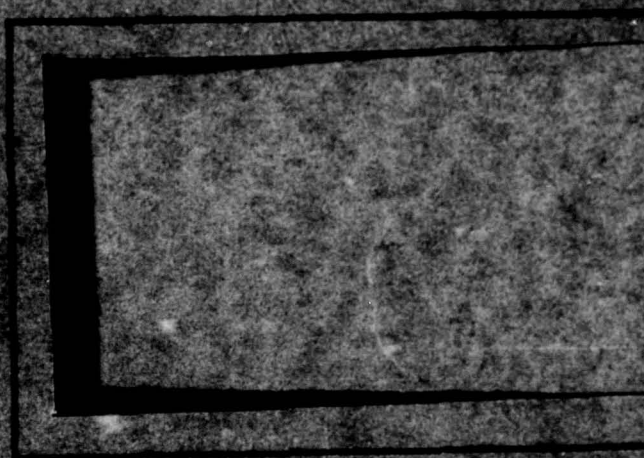


--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--

AD A04781



AIR UNIVERSITY  
UNITED STATES AIR FORCE



SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE

AD No. 71  
CDC FILE COPY

1

6 AN ALGORITHMIC APPROACH FOR REDUCING  
KINEMATICALLY SIMILAR  
DIFFERENTIAL SYSTEMS.

9 Master's THESIS

14 AFIT/GCS/MA/77D-1

10 Kevin/Dolan  
Captain USAF

Approved for public release; distribution unlimited

11 Dec 77

12 88p.

DDC  
RECEIVED  
DEC 20 1977  
E

1977

012 225

4B

AN ALGORITHMIC APPROACH FOR REDUCING  
KINEMATICALLY SIMILAR  
DIFFERENTIAL SYSTEMS

THESIS

Presented to the Faculty of the School of Engineering  
of the Air Force Institute of Technology  
Air University  
in Partial Fulfillment of the  
Requirements for the Degree of  
Master of Science

by

Kevin Dolan, B. S.

Captain USAF

Graduate Computer Science

December 1977

Approved for public release; distribution unlimited



## Preface

My studies in computer science have led me to believe that the primary duty of a computer scientist should be in search of new algorithms that will enhance computer efficiency. This thesis evolved in search of a new algorithm that could be used to reduce a differential system. It would not have been possible without the patience and imagination of Dr. John Jones, Jr., who pointed me in the right direction. The Air Force is indeed fortunate to have him as a team member.

This work was sponsored by the Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio. I would like to thank Major Eric Linberg, Dr. Dan Repperger of the Aerospace Medical Laboratory, Dr. Robert Craig of the Air Force Materials Laboratory, and Professors Constantine Houpis and John D'Azzo of the Department of Electrical Engineering for taking the time to review this thesis. The results of this paper should be of value to the United States Air Force and others who need solutions to the types of equations that I have considered.

I owe a special debt of gratitude to my wife and son who have been as much a part of my work at AFIT as myself.

APPROVED FOR	File Section <input checked="" type="checkbox"/>
IS	R ft Section <input type="checkbox"/>
NO	<input type="checkbox"/>
RECOMMENDED	
STANDARD	
DISTRIBUTION/AVAILABILITY CODES	
SPECIAL	
A	

## Contents

Preface .....	ii
List of Tables .....	iv
Abstract .....	v
I. Introduction .....	1
II. Solutions to Polynomial Equations .....	4
2.1. Solutions to Certain Polynomial Equations .....	4
2.2. A Fundamental Solution to Eq (2.1) .....	21
III. Reducibility Using Solutions of the Liapunov Equation ...	25
3.1. Reducibility of Eq (1.1) to Upper Triangular Form .....	25
3.2. Reducibility of Eq (1.1) Using a Solution of the Liapunov Equation .....	26
3.3. Reducibility of Time Dependent Forms of Eq(1.1). ..	29
3.4. Iterative Methods Used to Solve Eq (3.13) .....	37
IV. Reducibility Using Solutions of the Riccati Equation ....	49
4.1. Reducibility of Eq (1.1) for the Constant Case .	49
4.2. Reducibility of Time Dependent Forms of Eq(1.1). ..	56
4.3. Solutions to Eq (4.4) .....	62
Iterative Methods to Solve Eq (4.4) .....	68
V. Conclusion .....	76
Bibliography .....	77
Vita .....	79

List of Tables

<u>Table</u>		<u>Page</u>
I	Solution to Eq(3.13) by Iterative Method .....	45
II	Combinations of Characteristic Roots of R .....	63
III	Solutions to Eq(4.4) by Iterative Method .....	71
IV	Eq(4.110) to Solve Square Root of a Matrix .....	75

Abstract

Methods of reducing linear homogeneous differential matrix equations of the form  $\dot{\bar{X}} = A(t)\bar{X}$  by kinematically similar transformations are presented in this thesis. Classical equivalence and similarity relationships between matrices with elements belonging to the polynomial domain with coefficients belonging to the field of real numbers are obtained which give necessary and sufficient conditions for the existence of solutions to certain polynomial equations. Block reduction techniques using kinematically similar transformations lead to a discussion of iterative and non-iterative solution methods for the Liapunov matrix equation, Liapunov differential matrix equation, Riccati matrix equation, and the Riccati differential matrix equation.



AN ALGORITHMIC APPROACH FOR REDUCING  
KINEMATICALLY SIMILAR  
DIFFERENTIAL SYSTEMS

I. Introduction

The solution of differential equations of the form

$$\dot{\vec{X}}(t) = \tilde{A}(t) \vec{X}(t) \quad (1.1)$$

where  $\tilde{A}(t)$  is an  $n \times n$  matrix, has a wide variety of applications in the engineering sciences. In a computer environment solutions to Eq (1.1) can sometimes involve a time consuming iterative technique with questionable accuracy. Classical methods of reducing Eq (1.1) to a simpler system have generally been limited to the case where the matrix  $\tilde{A}$  is a constant matrix or has periodic or almost periodic elements. For the case where the matrix  $\tilde{A}$  has constant elements, reduction techniques have involved the computation of eigenvectors. These eigenvectors are sometimes difficult to compute when the matrix  $\tilde{A}$  has multiple characteristic roots and the process cannot always be accomplished in an efficient way on a computer. This thesis will develop a new algorithmic approach for reducing differential systems with matrices such as  $\tilde{A}$  in Eq (1.1) for both the constant and some variable cases so that the computation of the eigenvectors or a long iterative process is not required in order to obtain a solution. The reduction technique will allow one to reduce the matrix  $\tilde{A}$  by blocks either partially or completely under rather general conditions on  $\tilde{A}$ . This reduction of Eq (1.1) to a simpler but similar system is extremely advantageous in terms of the computer time and memory requirements



needed to solve large differential systems.

The idea of reducing a matrix to a simpler but similar form was presented by H. Loewy (Ref 1). He defined a relationship between two matrices whose elements are functions of a variable  $x$  as follows:

Definition 1.1. Let

$$P = (p_{ij}(x)), P^D = \left(\frac{d p_{ij}(x)}{dx}\right), (i, j = 1, 2, \dots, n) \quad (1.2)$$

then we may write  $A \stackrel{Lo}{\sim} B$  ( $A$  is similar to  $B$  in the sense of Loewy) if there exists a matrix  $P$  whose elements are differentiable functions of  $x$  and whose determinant does not vanish identically such that

$$B(x) = -P^D P^{-1}(x) + P(x) A(x) P^{-1}(x) \quad (1.3)$$

Further research in this area has been done by Coppel (Ref 2), and as presented by Berkey (Ref 3) it states that a differential system in the form of Eq (1.1) is kinematically similar to another equation

$$\dot{\overline{Y}}(t) = B(t) \overline{Y}(t) \quad (1.4)$$

of the same form, if there exists a continuously differentiable invertible matrix function  $P(t)$  satisfying the equation

$$\dot{P}(t) = A(t) P(t) - P(t) B(t) \quad (1.5)$$

which is bounded together with its inverse on the real line. The vector change of variable

$$\overline{X}(t) = P(t) \overline{Y}(t) \quad (1.6)$$

transforms Eq (1.1) into (1.4) so that they both have the same stability and boundedness properties.

In the process of developing a P matrix, as suggested by Loewy and Coppel, solutions of the Liapunov equation and the Riccati equation will be investigated. These equations by themselves have a wide variety of applications in the field of engineering sciences. Computational methods used to solve these equations can be found throughout the literature. Of particular note is work done by Leuthauser (Ref 4) which will be discussed in Chapter IV.

In this thesis capital letters denote matrices and matrix functions, while lower case letters represent scalars and scalar functions. T is the transpose symbol, and I is the identity matrix.

## II. Solutions to Polynomial Equations

The main purpose of this chapter is to show how the solutions to certain matrix equations with elements belonging to a polynomial domain  $F[z]$ , where  $F$  is the field of real numbers, implies the equivalency and similarity of pairs of matrices. The solutions of the matrix equations are in themselves a worthy topic and can be found in various works such as those of B. Porter (Ref 5), and as will be shown later in this chapter as a means to evaluate the fundamental matrix  $e^{At}$  where  $A$  is the  $n \times n$  matrix of the differential system given by Eq (1.1). This in turn will have a direct application to obtaining a fundamental solution to

$$\dot{\overline{X}}(t) = \tilde{A} \overline{X}(t) \quad (2.1)$$

where  $\tilde{A}$  has elements belonging to the field of real numbers.

### 2.1 Solutions to Certain Polynomial Equations

Definition 2.1. The matrix  $A$  is said to be similar to the matrix  $B$  if and only if there exists a non-singular matrix  $P$  such that  $P^{-1} A P = B$ .

Definition 2.2. The matrix  $A$  is equivalent to the matrix  $B$  if and only if there exists non-singular matrices  $P$  and  $Q$  such that  $P A Q = B$ .

Theorem 2.1. Let

$$A(z) X(z) = C(z) \quad (2.2)$$

where  $A$ ,  $X$ , and  $C$  are  $n \times n$  matrices having elements belonging to  $F[z]$ , then the following  $2n \times 2n$  matrices are equivalent:

$$\left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & I \end{array} \right], \quad \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & I \end{array} \right] \quad (2.3)$$

Proof. By definition 2.2 one must demonstrate that there exists a pair of matrices P and Q so that the pair of matrices given by (2.3) are equivalent. Let X(z) be a solution of Eq (2.2) and let

$$P = \left[ \begin{array}{c|c} I & -A(z)X(z) \\ \hline 0 & I \end{array} \right], \quad Q = \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right] \quad (2.4)$$

Using these matrices as required in definition 2.2 yields

$$\begin{aligned} \left[ \begin{array}{c|c} I & -A(z)X(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right] = \\ \left[ \begin{array}{c|c} A(z) & C(z) - A(z)X(z) \\ \hline 0 & I \end{array} \right] = \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & I \end{array} \right] \end{aligned} \quad (2.5)$$

since Eq (2.2) has a solution by hypothesis.

Theorem 2.2. If X(z) is a solution of Eq (2.2) then the following pair of  $2n \times 2n$  matrices are similar:

$$\left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & 0 \end{array} \right], \quad \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & 0 \end{array} \right] \quad (2.6)$$

Proof. By definition 2.1 one must produce a non-singular matrix P so that the pair of  $2n \times 2n$  matrices in (2.6) are similar. Let

$$P = \left[ \begin{array}{c|c} I & -X(z) \\ \hline 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & X(z) \\ \hline 0 & I \end{array} \right] \quad (2.7)$$

and using these matrices as required in definition 2.1 yields

$$\begin{bmatrix} I & X(z) \\ 0 & I \end{bmatrix} \begin{bmatrix} A(z) & C(z) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -X(z) \\ 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} A(z) & C(z) - A(z)X(z) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A(z) & 0 \\ 0 & 0 \end{bmatrix} \quad (2.8)$$

Theorem 2.3. Let  $X(z)$  be a solution of the matrix Eq (2.2) then the following equation must hold true:

$$\begin{bmatrix} I & X(z) \end{bmatrix} \begin{bmatrix} A(z) & C(z) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -X(z) \\ I \end{bmatrix} = (0) \quad (2.9)$$

Proof. Multiplication of the left-hand side of Eq (2.9) will yield the zero matrix since Eq (2.2) is assumed to have a solution  $X(z)$ .

Theorem 2.4. Let  $X(z)$  be a solution to Eq (2.2) then

$$\begin{bmatrix} A(z) & C(z) \end{bmatrix} \begin{bmatrix} -X(z) \\ I \end{bmatrix} = (0) \quad (2.10)$$

Proof. Multiplication of the left-hand side of Eq (2.10) will yield the zero matrix since Eq (2.2) is assumed to have a solution  $X(z)$ .

Before proceeding with theorem 2.5 it will be necessary to standardize some notation:

1) For polynomial elements of a matrix,  $a_{ii}(z) | a_{jj}(z)$ , where  $z$  is a real variable, means that the element  $a_{ii}(z)$  of matrix  $A$  divides into the element  $a_{jj}(z)$  and the quotient is also a polynomial in  $z$ .



2)  $(a_{ii}(z), a_{jj}(z))$  means the highest polynomial common factor of  $a_{ii}(z)$  and  $a_{jj}(z)$ . For example,

$$((z+1), (z+1)^2) = (z+1) \quad (2.11)$$

and also

$$((z+1), z) = 1 \quad (2.12)$$

Eq (2.12) implies that  $a_{ii}(z) = (z+1)$  is relatively prime to  $a_{jj}(z) = z$ . A useful result that will be used in conjunction with this notation is that if  $f(z)$ ,  $g(z)$  are polynomials in  $z$  and  $f(z)$ ,  $g(z)$  are relatively prime to each other, that is

$$(f(z), g(z)) = 1 \quad (2.13)$$

then there exist  $a(z)$  and  $b(z)$  which are polynomials in  $z$  such that

$$a(z)f(z) + b(z)g(z) = 1 \quad (2.14)$$

Theorem 2.5. If the pair of  $2n \times 2n$  matrices

$$\left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & A(z) \end{array} \right], \quad \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & A(z) \end{array} \right] \quad (2.15)$$

are equivalent then Eq (2.2) has a solution  $X(z)$  with polynomial elements in  $z$ .

Proof. Using the results of W. E. Roth (Ref 6) the equivalency of the pair of matrices of (2.15) implies the existence of non-singular matrices  $P$  and  $Q$  as required by definition 2.2 such that

$$P(z)A(z)Q(z) = \text{diag} \{a'_{11}(z), a'_{22}(z), \dots, a'_{\alpha\alpha}(z), 0, 0, \dots, 0\} \quad (2.16)$$

where  $a'_{ii}(z) | a'_{jj}(z)$  for  $i < j = 2, 3, \dots, \alpha$ . Using the same matrices  $P(z)$ ,

$Q(z)$ , and  $C(z)$  yields

$$P(z)C(z)Q(z) = C'(z) \quad (2.17)$$

where  $C'(z)$  has elements  $c'_{ij}(z)$  which are polynomials in  $z$  such that

$$c'_{ij}(z) = k_{ij}(z)(a'_{ii}(z), a'_{\alpha\alpha}(z)), (i,j=1,2,\dots,n) \quad (2.18)$$

where  $k_{ij}(z)$  is a constant or a polynomial in  $z$ .

Since  $(a'_{ii}(z), a'_{\alpha\alpha}(z)) = a'_{ii}(z)$ , Eq (2.18) reduces to

$$c'_{ij}(z) = k_{ij}(z) a'_{ii}(z), (i,j=1,2,\dots,n) \quad (2.19)$$

Thus an element by element solution to the equations

$$a'_{ii}(z)u_{ij}(z) = c'_{ij}(z), (i,j=1,2,\dots,n) \quad (2.20)$$

would have a solution

$$u_{ij}(z) = k_{ij}(z), (i,j=1,2,\dots,n) \quad (2.21)$$

In matrix notation the following equation would be valid from

Eq (2.20):

$$A'U = C' \quad (2.22)$$

But from Eqs (2.16) and (2.17) this is

$$[PAQ]U = PCQ \quad (2.23)$$

Multiplying Eq (2.23) on the left by  $P^{-1}$  and on the right by  $Q^{-1}$

yields

$$A[QUQ^{-1}] = C \quad (2.24)$$

which implies

$$X(z) = Q U Q^{-1} \quad (2.25)$$

is a solution to Eq (2.2).

Theorem 2.5 implies that by finding non-singular matrices P and Q that produce the diagonal matrix of Eq (2.16), one can then generate a matrix U by solving Eq (2.20) element-wise and thus produce a solution X(z) to Eq (2.2) by computing  $Q U Q^{-1}$  in Eq (2.25).

Example 2.1. Find X(z) in Eq (2.2) when

$$A(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^2+z & 0 \\ 0 & -z^2 & z \end{bmatrix}, \quad C(z) = \begin{bmatrix} z & 1-z^2 & z^2 \\ z^3-z & z^2+z & z^3-z \\ z^2+z & z^3-z^2-z & z^2 \end{bmatrix} \quad (2.26)$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad (2.27)$$

Eq (2.16) yields

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^2+z \end{bmatrix} = P A(z) Q \quad (2.28)$$

Eq (2.17) yields

$$C' = \begin{bmatrix} z & z^2 & 1-2z^2 \\ z^3+z^2 & z^3+z^2-z & -z^2+z \\ z^3-z & z^3-z & -z^3+z^2+2z \end{bmatrix} = P C(z) Q \quad (2.29)$$

Now constructing  $u_{ij}(z)$  from Eq (2.20) yields

$$\left. \begin{array}{lll} u_{11} = z & u_{12} = z^2 & u_{13} = 1-2z^2 \\ u_{21} = z^2+z & u_{22} = z^2+z-1 & u_{23} = -z+1 \\ u_{31} = z-1 & u_{32} = z-1 & u_{33} = -z+2 \end{array} \right\} \quad (2.30)$$

Eq (2.25) will yield

$$X(z) = \begin{bmatrix} z & 1-z^2 & z^2 \\ z-1 & 1 & z-1 \\ z^2+1 & z^2-1 & z^2 \end{bmatrix} = Q U Q^{-1} \quad (2.31)$$

It should be noted that when Eq (2.2) has rational polynomials as elements, it can be reduced to the case where the elements are polynomials by multiplying Eq (2.2) by the least common multiple of the denominators of  $A(z)$  and  $C(z)$ .

Example 2.2. If Eq (2.2) were of the form

$$X(z) = \begin{bmatrix} \frac{z^2+z}{z^2-1} & \frac{z}{z+1} \\ \frac{z+5}{z^2+2z+1} & z \end{bmatrix} = \begin{bmatrix} \frac{z}{z-1} & \frac{z^2}{z+1} \\ \frac{z}{(z+1)^2} & \frac{2z}{z-1} \end{bmatrix} \quad (2.32)$$

then it could be reduced to a form of only polynomial elements by multiplying  $A(z)$  and  $C(z)$  by  $(z-1)(z+1)^2$  which is the least common multiple of the denominator terms of the elements.

The next theorem will extend the ideas presented to the solution of a differential equation whose coefficient matrix has elements belonging to the polynomial domain  $F[z]$ .

Theorem 2.6. If  $X(z)$  is a solution of the differential equation

$$\dot{X}(z) = A(z) X(z) - C(z) \quad (2.33)$$

then the following matrices are equivalent

$$\left[ \begin{array}{c|c} A(z) & C(z) + \dot{X}(z) \\ \hline 0 & I \end{array} \right], \quad \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & I \end{array} \right] \quad (2.34)$$

Proof. To satisfy definition 2.2, let

$$P = \left[ \begin{array}{c|c} I & -A(z)X(z) \\ \hline 0 & I \end{array} \right], \quad Q = \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right] \quad (2.35)$$

and thus

$$\begin{aligned} & \left[ \begin{array}{c|c} I & -A(z)X(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} A(z) & C(z) + \dot{X}(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array} \right] = \\ & \left[ \begin{array}{c|c} A(z) & C(z) + \dot{X}(z) - A(z)X(z) \\ \hline 0 & I \end{array} \right] = \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & I \end{array} \right] \quad (2.36) \end{aligned}$$

The next theorems use an approach similar to the previous theorems in order to solve a matrix equation of the form

$$A(z)X(z) - X(z)A(z) = C(z) \quad (2.37)$$

where  $A(z)$ ,  $C(z)$ , and  $X(z)$  are  $n \times n$  matrices having elements belonging to the polynomial domain  $F[z]$ .

Theorem 2.7. If  $X(z)$  is a solution of Eq (2.37) then the pair of matrices given in (2.15) are similar.



Proof. Let

$$P = \left[ \begin{array}{c|c} I & -X(z) \\ \hline 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & X(z) \\ \hline 0 & I \end{array} \right] \quad (2.38)$$

in order to satisfy definition 2.1. Thus

$$\left[ \begin{array}{c|c} I & X(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & A(z) \end{array} \right] \left[ \begin{array}{c|c} I & -X(z) \\ \hline 0 & I \end{array} \right] = \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & A(z) \end{array} \right] \quad (2.39)$$

provided  $X(z)$  is a solution to Eq (2.37).

Theorem 2.8. If  $X(z)$  is a solution of the differential equation

$$\dot{X}(z) = A(z)X(z) - X(z)A(z) - C(z) \quad (2.40)$$

then the following pair of matrices are similar

$$\left[ \begin{array}{c|c} A(z) & C(z) + \dot{X}(z) \\ \hline 0 & A(z) \end{array} \right], \quad \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & A(z) \end{array} \right] \quad (2.41)$$

Proof. Let

$$P = \left[ \begin{array}{c|c} I & -X(z) \\ \hline 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & X(z) \\ \hline 0 & I \end{array} \right] \quad (2.42)$$

in order to satisfy definition 2.1. Thus

$$\left[ \begin{array}{c|c} I & X(z) \\ \hline 0 & I \end{array} \right] \left[ \begin{array}{c|c} A(z) & C(z) + \dot{X}(z) \\ \hline 0 & A(z) \end{array} \right] \left[ \begin{array}{c|c} I & -X(z) \\ \hline 0 & I \end{array} \right] =$$

$$\left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & A(z) \end{array} \right] \quad (2.43)$$

with  $X(z)$  a solution of Eq (2.40).

The next theorem is given in Nerring (Ref 7:115) and will be needed to prove theorem 2.10.

Theorem 2.9. If the  $(\text{trace of } C(z)) = 0$  then

$$(\text{trace}[PC(z)P^{-1}]) = 0 \quad (2.44)$$

Theorem 2.10. If the pair of matrices in (2.15) are equivalent then Eq (2.37) has a solution  $X(z)$  provided the following conditions hold:

- 1)  $\text{Trace } [C(z)] = 0$
- 2) There exists a non-singular matrix  $P$  such that  $P^{-1}AP$  equals a diagonal matrix
- 3) For all  $\alpha$  and  $\beta$ ,  $\alpha \neq \beta$  the invariant factors  $a_{\alpha\alpha}(z)$ ,  $a_{\beta\beta}(z)$  of  $P^{-1}A(z)P$  must have the property that  $a_{\alpha\alpha}(z) \neq a_{\beta\beta}(z)$ .

Proof. Let  $P A(z) P^{-1} = \text{diag } \{a_{11}(z), a_{22}(z), \dots, a_{\beta\beta}(z), 0, 0, \dots, 0\}$  where  $a_{ii}(z) \nmid a_{jj}(z)$ ,  $(j=2, \dots, \beta)$  and  $a_{ii}(z) \neq a_{jj}(z)$  to satisfy condition 3. Since  $\text{Trace } [C(z)] = 0$  then  $\text{Trace } [PC(z)P^{-1}] = 0$  by theorem 2.9. Using the results of Roth (Ref 6) on equivalent matrices the following matrix equations can be formed:

$$\left[ \begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array} \right] \left[ \begin{array}{c|c} A(z) & C(z) \\ \hline 0 & A(z) \end{array} \right] \left[ \begin{array}{c|c} P^{-1} & 0 \\ \hline 0 & P^{-1} \end{array} \right] = \left[ \begin{array}{c|c} PA(z)P^{-1} & PC(z)P^{-1} \\ \hline 0 & PA(z)P^{-1} \end{array} \right]$$

and

$$\left[ \begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array} \right] \left[ \begin{array}{c|c} A(z) & 0 \\ \hline 0 & A(z) \end{array} \right] \left[ \begin{array}{c|c} P^{-1} & 0 \\ \hline 0 & P^{-1} \end{array} \right] = \left[ \begin{array}{c|c} PA(z)P^{-1} & 0 \\ \hline 0 & PA(z)P^{-1} \end{array} \right]$$

(2.45)

where  $PA(z)P^{-1} = A'(z)$  such that  $a'_{ij} \in (A'(z))$  and

$PC(z)P^{-1} = C'(z)$  such that  $c'_{ij} \in (C'(z))$ .

Forming the matrix M below and using the results of Roth (Ref 6)

$$\left[ \begin{array}{c|c} \begin{array}{cccc} a'_{11} & 0 & \dots & 0 \\ 0 & a'_{22} & 0 & \dots \\ 0 & \dots & a'_{\beta\beta} & 0 \\ 0 & \dots & 0 & 0 \end{array} & \begin{array}{ccc} c'_{11} & \dots & c'_{1\beta} & \dots & c'_{1n} \\ c'_{\beta 1} & \dots & c'_{\beta\beta} & \dots & c'_{\beta n} \\ c'_{n1} & \dots & c'_{n\beta} & \dots & c'_{nn} \end{array} \\ \hline \begin{array}{cccc} 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & \dots & 0 \end{array} & \begin{array}{cccc} a'_{11} & 0 & \dots & 0 \\ 0 & a'_{22} & 0 & \dots \\ 0 & \dots & a'_{\beta\beta} & 0 \\ 0 & \dots & 0 & 0 \end{array} \end{array} \right]$$

(2.46)

the elements  $c'_{ij}$  are then the highest common factors of  $a'_{ii}(z)$  and  $a'_{jj}(z)$ , thus

$$c'_{ij}(z) = k_{ij}(z)(a'_{ii}(z), a'_{jj}(z)), (i, j=1, 2, \dots, n) \quad (2.47)$$

where  $k_{ij}(z)$  is a constant or a polynomial in  $z$ . Therefore, to solve the equations

$$a'_{\alpha\alpha}(z) u_{\alpha\beta}(z) - u_{\alpha\beta}(z) a'_{\beta\beta}(z) = c'_{\alpha\beta}(z) \quad (2.48)$$

where  $u_{\alpha\beta}$  are unknown polynomial functions, one must investigate three possibilities:

Case 1.  $\alpha < \beta$

Case 2.  $\alpha > \beta$

Case 3.  $\alpha = \beta$

Case 1. For  $\alpha < \beta$  then  $a'_{\alpha\alpha}(z) | a'_{\beta\beta}(z)$  and Eq (2.48) can be written as

$$a_{\alpha\alpha}(z) u_{\alpha\beta}(z) - u_{\alpha\beta}(z) [l_{\alpha\beta}(z) a_{\alpha\alpha}(z)] = k_{\alpha\beta}(z) a_{\alpha\alpha}(z) \quad (2.49)$$

where  $l_{\alpha\beta} = \frac{a'_{\beta\beta}(z)}{a'_{\alpha\alpha}(z)}$ . Therefore

$$u_{\alpha\beta}(z) - u_{\alpha\beta}(z) l_{\alpha\beta}(z) = k_{\alpha\beta}(z) \quad (2.50)$$

$$u_{\alpha\beta}(z) = \frac{k_{\alpha\beta}(z)}{1 - l_{\alpha\beta}(z)} \quad (2.51)$$

which is a ratio of polynomials.

Case 2. For  $\alpha > \beta$  then  $a_{\beta\beta} | a_{\alpha\alpha}(z)$  and Eq (2.48) can be written as

$$[l_{\alpha\beta}(z) a_{\beta\beta}(z)] u_{\alpha\beta}(z) - u_{\alpha\beta}(z) a_{\beta\beta}(z) = k_{\alpha\beta}(z) a_{\beta\beta}(z) \quad (2.52)$$

where  $l_{\alpha\beta}(z) = \frac{a_{\alpha\alpha}(z)}{a_{\beta\beta}(z)}$ . Thus

$$u_{\alpha\beta}(z) = \frac{k_{\alpha\beta}(z)}{l_{\alpha\beta}(z) - 1} \quad (2.53)$$

Case 3. For  $\alpha = \beta$  then  $c_{\alpha\beta} = 0$  by condition 1 and the solution  $u_{\alpha\beta}$  is any arbitrary polynomial in  $z$ .

The elements  $u_{\alpha\beta}$  which form a matrix  $U$  can be used to solve Eq (2.37) since

$$A'U - UA' = C' \quad (2.54)$$

or in other words

$$(PA(z)P^{-1})U - U(PA(z)P^{-1}) = PC(z)P^{-1} \quad (2.55)$$

Multiplying Eq (2.55) on the left by  $P^{-1}$  and on the right by  $P$  yields

$$A(z)(P^{-1}UP) - (P^{-1}UP)A(z) = C(z) \quad (2.56)$$

Therefore a solution to Eq (2.37) would be given by

$$X(z) = P^{-1}UP \quad (2.57)$$

Example 2.3. Find  $X(z)$  in Eq (2.37) with

$$A(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^2+z & 0 \\ 0 & -z^2 & z \end{bmatrix}, \quad C(z) = \begin{bmatrix} 0 & z^4-z^2-z+1 & z^2-z^3 \\ z^4+z^3+z^2 & z^3 & z^3 \\ z-z^4-1 & z^2 & z^3 \end{bmatrix} \quad (2.58)$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad (2.59)$$



Forming the matrix M of Eq (2.46) yields

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & z^2 - z^3 & z^4 + z^3 - 2z^2 - z + 1 \\ 0 & z & 0 & z^3 - z^2 + z - 1 & 0 & z^3 + z^2 \\ 0 & 0 & z + z^2 & z^4 + z^3 - z^2 & z^3 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & z + z^2 \end{array} \right] \quad (2.60)$$

Solving  $u_{\alpha\beta}$  from Eq (2.48) yields

$$\left. \begin{aligned} u_{11} &= \text{arbitrary}; \quad u_{12} = z^2; \quad u_{13} = 1 - z^2 \\ u_{21} &= z^2 + 1; \quad u_{22} = \text{arbitrary}; \quad u_{23} = -1 - z \\ u_{31} &= z^2; \quad u_{32} = z; \quad u_{33} = \text{arbitrary} \end{aligned} \right\} \quad (2.61)$$

Solving Eq (2.57) for  $X(z)$  yields

$$X(z) = \begin{bmatrix} 0 & 1 & z^2 \\ z^2 & z & z \\ 1 & 1-2z & -z \end{bmatrix} = P^{-1} U P \quad (2.62)$$

It should be noted that in the examples generated  $u_{\alpha\beta}$  are polynomials in  $z$  and not the more restrictive case implied by Eqs (2.51) and (2.53). It is possible that such is always the case, but a proof could not be generated to substantiate this.

The next example will show a direct application of W. E. Roth (Ref 6) which uses a similar technique to solve

$$AX - XA = C \quad (2.63)$$

where A and C belong to the field of real numbers.

Example 2.4.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 2 \\ -1 & 2 \end{bmatrix} \quad (2.64)$$

From Roth's lemma (Ref 6) if the matrices of Eq (2.15) are similar then the following matrices are equivalent with polynomial elements in  $x$  with real coefficients:

$$\left[ \begin{array}{c|c} (A-xI) & C \\ \hline 0 & (A-xI) \end{array} \right], \quad \left[ \begin{array}{c|c} (A-xI) & 0 \\ \hline 0 & (A-xI) \end{array} \right] \quad (2.65)$$

According to Roth this then implies the existence of matrices  $X$  and  $Y$  such that

$$(A-xI)X - Y(A-xI) = C \quad (2.66)$$

Finding matrices  $P$  and  $Q$  for  $(A-xI)$  as in theorem 2.5 yields

$$P = \left[ \begin{array}{c|c} 0 & 1 \\ \hline 1 & x-1 \end{array} \right], \quad Q = \left[ \begin{array}{c|c} 1 & x \\ \hline 0 & 1 \end{array} \right] \quad (2.67)$$

Thus for the example

$$P(A-xI)Q = \begin{bmatrix} 1 & 0 \\ 0 & x-x^2 \end{bmatrix} = A' \quad (2.68)$$

$$P C Q = \begin{bmatrix} -1 & 2-x \\ -x-1 & -x^2+x \end{bmatrix} = C' \quad (2.69)$$

Forming the matrix  $M$  in Eq (2.46) yields

$$M = \left[ \begin{array}{cc|cc} 1 & 0 & -1 & 2-x \\ 0 & x-x^2 & -x-1 & -x^2+x \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x-x^2 \end{array} \right] \quad (2.70)$$

Then as in theorem 2.10 solving an equation of the form

$$A'U - VA' = C' \quad (2.71)$$

will lead to a solution of Eq (2.66) with

$$X(x) = Q U Q^{-1} \quad (2.72)$$

$$Y(x) = P^{-1}VP \quad (2.73)$$

where

$$X = X_0 + X_1x + \dots + X_px^p \quad (2.74)$$

$$Y = Y_0 + Y_1x + \dots + Y_qx^q \quad (2.75)$$

such that  $X_i$  and  $Y_j$  are  $n \times n$  matrices with elements belonging to the field  $F$ . Thus using Eq (2.71) to solve for  $U$  and  $V$  element by element for the example yields

$$\begin{aligned} u_{11} &= 0; v_{11} = 1; & u_{12} &= (2-x); & v_{12} &= 0 \\ u_{21} &= 0; v_{21} = 0; & u_{22} &= 1; & v_{22} &= 0 \end{aligned} \quad (2.76)$$

Therefore

$$U = \begin{bmatrix} 0 & 2-x \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -x \\ 0 & 1 \end{bmatrix} \quad (2.77)$$

Applying Eqs (2.72) and (2.73) yields

$$X(x) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} = Y(x) \quad (2.79)$$

With Eq (2.79) as a solution to Eq (2.66), a solution to Eq (2.63) is

$$X = \sum_{i=0}^p A^i Y_i \quad (2.80)$$

or

$$X = \sum_{i=0}^p X_i B^i \quad (2.81)$$

where  $A^0 = B^0 = I$

Thus using Eqs (2.80) or (2.81) will yield

$$X = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad (2.82)$$

as a solution to Eq (2.63).

Example 2.4 shows that two solutions may be readily computed for Eq (2.63). The next theorem could prove useful if two such solutions are available.

Theorem 2.11. If  $X_1$  and  $X_2$  are solutions to Eq (2.63) then  $X_1 - X_2$  is a solution to

$$AW = WA \quad (2.83)$$

Proof. Let

$$AX_1 - X_1A = C \quad (2.84)$$

also

$$AX_2 - X_2A = C \quad (2.85)$$

Subtracting Eq (2.85) from (2.84) yields

$$A(X_1 - X_2) - (X_1 - X_2)A = 0 \quad (2.86)$$

By letting

$$W = X_1 - X_2 \quad (2.87)$$

a solution to Eq (2.83) is found.

## 2.2 A Fundamental Solution to Eq (2.1)

This section will show how the solution of polynomial matrix equations can be used to compute a solution to Eq (2.1).

Theorem 2.12 (Bronson (Ref 8)). The matrix system in Eq (2.1) with  $\overline{X}(t_0) = \overline{k}$  has the solution

$$\overline{X}(t) = [e^{\tilde{A}(t-t_0)}] \overline{k} \quad (2.88)$$

Theorem 2.13 (Bronson (Ref 8)). If  $\tilde{A}$  is a matrix having  $n$  rows and  $n$  columns then

$$e^{\tilde{A}t} = \alpha_{n-1}(t)\tilde{A}^{n-1}t^{n-1} + \alpha_{n-2}(t)\tilde{A}^{n-2}t^{n-2} + \dots + \alpha_1(t)\tilde{A}t + \alpha_0(t)I \quad (2.89)$$

where  $\alpha_0(t), \alpha_1(t), \dots, \alpha_{n-1}(t)$  are polynomials in  $t$  which must be determined for a given  $\tilde{A}$  in Eq (2.1).

Theorem 2.14 (Bronson (Ref 8)). Let  $\tilde{A}$  be as in theorem 2.13 and define

$$\gamma(\lambda) = \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \dots + \alpha_1\lambda + \alpha_0 \quad (2.90)$$

then if  $\lambda_i$  is an eigenvalue of  $\tilde{A}t$

$$e^{\gamma_i} = \gamma(\lambda_i) \quad (2.91)$$



Furthermore, if  $\lambda_1$  is an eigenvalue of multiplicity  $k$ ,  $k > 1$  then the following equations are also valid:

$$\begin{aligned}
 e^{\lambda_1} &= \left. \frac{d\gamma(\lambda)}{d\lambda} \right|_{\lambda=\lambda_1} \\
 e^{\lambda_1} &= \left. \frac{d^2\gamma(\lambda)}{d\lambda^2} \right|_{\lambda=\lambda_1} \\
 e^{\lambda_1} &= \left. \frac{d^{k-1}\gamma(\lambda)}{d\lambda^{k-1}} \right|_{\lambda=\lambda_1}
 \end{aligned} \tag{2.92}$$

Example 2.5. Find a solution to

$$\dot{\vec{X}}(t) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \vec{X}(t) \tag{2.93}$$

with  $\vec{X}(t_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Applying theorems (2.13) and (2.14) in order to find  $e^{\tilde{A}t}$  produces the characteristic roots of  $\tilde{A}t$  as:  $-t$ ,  $5t$

By applying Eq (2.91) yields:

$$\gamma(-t) = \alpha_1(t) \cdot (-t) + \alpha_0(t) = e^{-t} \tag{2.94}$$

$$\gamma(5t) = \alpha_1(t) \cdot (5t) + \alpha_0(t) = e^{5t} \tag{2.95}$$

Transforming Eqs (2.94) and (2.95) into matrix form yields:

$$\begin{bmatrix} -t & 1 \\ 5t & 1 \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_0(t) \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{5t} \end{bmatrix} \tag{2.96}$$

Eq (2.96) is now in the form of Eq (2.2) and theorem 2.5 can be used to solve for  $\alpha_0$  and  $\alpha_1$  with

$$P = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2.97)$$

and applying Eq (2.16) yields

$$A' = \begin{bmatrix} 6 & 0 \\ 0 & -6t \end{bmatrix} \quad (2.98)$$

Similarly applying Eq (2.17) by expanding the matrix  $C(t)$  of Eq (2.96) to square dimensions yields

$$C' = \begin{bmatrix} (5e^{-t}+e^{5t}) & (5e^{-t}+e^{5t}) \\ (e^{-t}-e^{5t}) & (e^{-t}-e^{5t}) \end{bmatrix} \quad (2.99)$$

Constructing the matrix  $U$  from Eq (2.20) yields

$$U = \begin{bmatrix} \frac{5e^{-t}-e^{5t}}{+6} & \frac{5e^{-t}-e^{5t}}{+6} \\ \frac{e^{-t}-e^{5t}}{-6t} & \frac{e^{-t}-e^{5t}}{-6t} \end{bmatrix} \quad (2.100)$$

then solving Eq (2.25) implies

$$\begin{bmatrix} \alpha_1(t) \\ \alpha_0(t) \end{bmatrix} = \begin{bmatrix} \frac{e^{5t}-e^{-t}}{6t} \\ \frac{5e^{-t}+e^{5t}}{6} \end{bmatrix} \quad (2.101)$$

Thus substituting  $\alpha_0$  and  $\alpha_1$  into Eq (2.89) yields

$$e^{\tilde{A}t} = \frac{1}{6} \begin{bmatrix} 2e^{5t}+4e^{-t} & 2e^{5t}-2e^{-t} \\ 4e^{5t}-4e^{-t} & 4e^{5t}+2e^{-t} \end{bmatrix} \quad (2.102)$$

Then using theorem 2.12, a solution to Eq (2.93) is

$$\overrightarrow{x(t)} = \frac{1}{6} \begin{bmatrix} 4e^{5t} + 2e^{-t} \\ 8e^{5t} - 2e^{-t} \end{bmatrix} \quad (2.103)$$

By taking advantage of equivalency and similarity relationships, this chapter developed methods to solve equations with polynomials as elements. Other methods are of course available, in particular the works of M.T. McClellan should be referenced. (Ref 9), (Ref 10).

### III. Reducibility Using Solutions of the Liapunov Equation

This chapter will apply the ideas of kinematic similarity presented in Chapter I to specific forms of

$$\dot{\overline{X}}(t) = \tilde{A}(t) \overline{X}(t) \quad (1.1)$$

The first section will show how reducibility of a differential system to a kinematically similar differential system is applicable and includes the classical reduction method. The second section will apply the reduction techniques to Eq (1.1) for the constant case by solving the Liapunov Equation. The third section will extend the ideas to the time dependent case, and the final section of this chapter will discuss possible methods that could be used to solve the Liapunov Equation.

#### 3.1 Reducibility of Eq (1.1) to Upper Triangular Form

Theorem 3.1 (Noble (Ref 11)). If  $\tilde{A}$  is a  $n \times n$  constant matrix with eigenvalues  $\lambda_1 \dots \lambda_n$  (which need not be distinct) then there exists a non-singular matrix  $P$  such that

$$P^{-1} \tilde{A} P = U \quad (3.1)$$

where  $U$  is an upper triangular matrix with  $\lambda_1 \dots \lambda_n$  along the diagonal.

Theorem 3.2. If  $\tilde{A}$  satisfies the conditions of theorem 3.1 then the differential system of Eq (1.1) is kinematically similar to

$$\dot{\overline{Y}}(t) = U \overline{Y}(t) \quad (3.2)$$

Proof. Let

$$\overline{X}(t) = P \overline{Y}(t) \quad (3.3)$$

where  $P$  is as defined in theorem 3.1.

Differentiating Eq (3.3) and the substitution into Eq (1.1) yields

$$\dot{\overline{X}}(t) = \tilde{A} P \overline{Y}(t) = P \dot{\overline{Y}}(t) \quad (3.5)$$

Since  $P^{-1}$  exists, Eq (3.5) transforms to

$$\dot{\overline{Y}}(t) = [P^{-1} \tilde{A} P] \overline{Y}(t) \quad (3.6)$$

which by theorem 3.1 is the desired result.

Example 3.1. Reduce

$$\dot{\overline{X}}(t) = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix} \overline{X}(t) \quad (3.7)$$

The matrix  $P$  of theorem 3.1 from Noble (Ref 11) is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad (3.8)$$

Using Eq (3.6), the system of Eq (3.7) reduces to

$$\dot{\overline{Y}}(t) = \begin{bmatrix} -2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix} \overline{Y}(t) \quad (3.9)$$

### 3.2 Reducibility of Eq (1.1) Using a Solution of the Liapunov Equation

Theorem 3.3. The differential system of Eq (1.1) in the form

$$\dot{\overline{X}}(t) = \begin{bmatrix} A & -C \\ 0 & B \end{bmatrix} \overline{X}(t) \quad (3.10)$$

where  $A$ ,  $B$ , and  $C$  are  $n \times n$  matrices with elements belonging to the field



of real numbers, is kinematically similar to

$$\dot{\overrightarrow{Y}}(t) = \left[ \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \overrightarrow{Y}(t) \quad (3.11)$$

by the transformation

$$\overrightarrow{X}(t) = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \overrightarrow{Y}(t) \quad (3.12)$$

where  $X$  is a solution to the equation

$$AX - XB = C \quad (3.13)$$

Proof. Let

$$\overrightarrow{X}(t) = P \overrightarrow{Y}(t) \quad (3.3)$$

where

$$P = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & -X \\ \hline 0 & I \end{array} \right] \quad (3.14)$$

Differentiating Eq (3.3) yields

$$\dot{\overrightarrow{X}}(t) = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \dot{\overrightarrow{Y}}(t) \quad (3.15)$$

Substituting Eq (3.3) into Eq (3.10) and equating with Eq (3.15) yields

$$\dot{\overrightarrow{X}}(t) = \left[ \begin{array}{c|c} A & -C \\ \hline 0 & B \end{array} \right] \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \overrightarrow{Y}(t) = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \dot{\overrightarrow{Y}}(t) \quad (3.16)$$

Multiplying Eq (3.16) on the left by  $P^{-1}$  yields

$$\overrightarrow{\dot{Y}(t)} = \left[ \begin{array}{c|c} A & AX - C - XB \\ \hline 0 & B \end{array} \right] \overrightarrow{Y(t)} \quad (3.17)$$

which reduces to Eq (3.11) with X as a solution to Eq (3.13).

Example 3.2. Reduce

$$\overrightarrow{\dot{X}(t)} = \left[ \begin{array}{cc|cc} 1 & 0 & -1 & -3 \\ 1 & 0 & -1 & -2 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array} \right] \overrightarrow{X(t)} \quad (3.18)$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}, \quad -C = \begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix} \quad (3.19)$$

a solution to Eq (3.13) is

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (3.20)$$

thus using the matrices of (3.14)

$$P = \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right], \quad P^{-1} = \left[ \begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (3.21)$$

the substitution of X in Eq (3.3) reduces Eq (3.18) to

$$\overrightarrow{\dot{Y}(t)} = \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array} \right] \overrightarrow{Y(t)} \quad (3.22)$$

A slightly different version of theorem 3.3 that will be used in Chapter IV

is as follows:

Theorem 3.4. The differential system of Eq (1.1) in the form

$$\dot{\overline{X}}(t) = \left[ \begin{array}{c|c} B & 0 \\ \hline -C & A \end{array} \right] \overline{X}(t) \quad (3.23)$$

is kinematically similar to

$$\dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right] \overline{Y}(t) \quad (3.24)$$

by the transformation

$$\overline{X}(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline X & I \end{array} \right] \overline{Y}(t) \quad (3.25)$$

where X is a solution to Eq (3.13).

Proof. Let

$$P = \left[ \begin{array}{c|c} I & 0 \\ \hline X & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & 0 \\ \hline -X & I \end{array} \right] \quad (3.26)$$

Differentiating Eq (3.25) and substitution into Eq (3.23) yields

$$\left[ \begin{array}{c|c} B & 0 \\ \hline -C & A \end{array} \right] \left[ \begin{array}{c|c} I & 0 \\ \hline X & I \end{array} \right] \dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline X & I \end{array} \right] \dot{\overline{Y}}(t) \quad (3.27)$$

Multiplication of Eq (3.27) by  $P^{-1}$  reduces Eq (3.23) to Eq (3.24) with X as a solution to Eq (3.13).

### 3.3 Reducibility of Time Dependent Forms of Eq (1.1)

Theorem 3.5. The differential system of Eq (1.1) in the form

$$\dot{\overline{X}}(t) = \left[ \begin{array}{c|c} A & -C(t) \\ \hline 0 & B \end{array} \right] \overline{X}(t) \quad (3.28)$$

where  $C(t)$  is an  $n \times n$  matrix whose elements are continuous real functions

of a real variable and A, B are as defined in theorem 3.3, is kinematically similar to

$$\dot{\overrightarrow{Y}}(t) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \overrightarrow{Y}(t) \quad (3.29)$$

by the transformation

$$\overrightarrow{X}(t) = \begin{bmatrix} I & X(t) \\ 0 & I \end{bmatrix} \overrightarrow{Y}(t) \quad (3.30)$$

where  $X(t)$  is a solution to

$$\dot{X}(t) = AX(t) - X(t)B - C(t) \quad (3.31)$$

Proof. Let

$$\overrightarrow{X}(t) = P \overrightarrow{Y}(t) \quad (3.3)$$

where

$$P = \begin{bmatrix} I & X(t) \\ 0 & I \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} I & -X(t) \\ 0 & I \end{bmatrix} \quad (3.32)$$

Differentiating Eq (3.3) yields

$$\dot{\overrightarrow{X}}(t) = \begin{bmatrix} I & X(t) \\ 0 & I \end{bmatrix} \dot{\overrightarrow{Y}}(t) + \begin{bmatrix} 0 & \dot{X}(t) \\ 0 & 0 \end{bmatrix} \overrightarrow{Y}(t) \quad (3.33)$$

Substituting Eq (3.3) into Eq (3.28) and equating with Eq (3.33) yields

$$\begin{bmatrix} A & -C(t) \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X(t) \\ 0 & I \end{bmatrix} \overrightarrow{Y}(t) = \begin{bmatrix} I & X(t) \\ 0 & I \end{bmatrix} \dot{\overrightarrow{Y}}(t) + \begin{bmatrix} 0 & \dot{X}(t) \\ 0 & 0 \end{bmatrix} \overrightarrow{Y}(t) \quad (3.34)$$

Multiplication and then the addition of factors of  $\overrightarrow{Y}(t)$  yields

$$\left[ \begin{array}{c|c} A & AX(t) - C(t) - \dot{X}(t) \\ \hline 0 & B \end{array} \right] \overrightarrow{Y(t)} = \left[ \begin{array}{c|c} I & X(t) \\ \hline 0 & I \end{array} \right] \overrightarrow{\dot{Y}(t)} \quad (3.35)$$

Multiplication of Eq (3.35) on the left by  $P^{-1}(t)$  yields

$$\overrightarrow{\dot{Y}(t)} = \left[ \begin{array}{c|c} A & AX(t) - X(t)B - C(t) - \dot{X}(t) \\ \hline 0 & B \end{array} \right] \overrightarrow{Y(t)} \quad (3.36)$$

which is the desired result with  $X(t)$  a solution to Eq (3.31).

The advantage of using this reduction technique can probably be best seen with an example that will first solve Eq (1.1) by a classical method and then by theorem 3.5.

Example 3.3.

$$\overrightarrow{\dot{X}(t)} = \begin{bmatrix} 1 & \sin t \\ 0 & 1 \end{bmatrix} \overrightarrow{X(t)} ; \quad X \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (3.37)$$

Eq (3.37) may be written in the following form:

$$\dot{x}_1(t) = x_1(t) + x_2(t) \sin t \quad (3.38)$$

$$\dot{x}_2(t) = x_2(t) \quad (3.39)$$

Solving for  $x_2(t)$  yields

$$\int_{t_0}^t \frac{\dot{x}_2(\tau)}{x_2(\tau)} d\tau = \int_{t_0}^t d\tau \quad (3.40)$$

Integrating Eq (3.40) and solving for  $x_2(t)$  yields

$$x_2(t) = x_2(t_0)e^{t-t_0} = e^t \quad (3.41)$$

Substituting this result in Eq (3.38) yields

$$\dot{x}_1(t) - x_1(t) = x_2(t_0)e^{t-t_0} \sin t \quad (3.42)$$



using  $e^{-t}$  as an integrating factor in Eq (3.42) yeilds

$$\int_{t_0}^t e^{-\tau} x_1(\tau) d\tau = \int_{t_0}^t x_2(t_0) e^{-t_0} \sin \tau d\tau \quad (3.43)$$

Integrating both sides and solving for  $x_1(t)$  yields

$$x_1(t) = 2e^t - e^t \cos t \quad (3.44)$$

Thus

$$\overrightarrow{X}(t) = \begin{bmatrix} 2e^t - e^t \cos t \\ e^t \end{bmatrix} \quad (3.45)$$

The same example using theorem 3.5 would be solved as follows: Let

$$A=1; B=1; -C(t) = \sin t \quad (3.46)$$

Solving Eq (3.31) for  $X(t)$  yields

$$\dot{X}(t) = \sin t \quad (3.47)$$

Eq (3.47) implies that

$$X(t) = -\cos t \quad (3.48)$$

except for a constant, and thus by theorem 3.5 Eq (3.37) will reduce to

$$\dot{\overrightarrow{Y}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overrightarrow{Y}(t) \quad (3.49)$$

with  $Y(t_0) = P^{-1}(t_0) X(t_0)$ . Thus

$$\begin{bmatrix} y_1(t_0) \\ y_2(t_0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (3.50)$$

A solution to Eq (3.49) is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1(t_0)e^t \\ y_2(t_0)e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} \quad (3.51)$$

Then using Eq (3.3) yields

$$\vec{x}(t) = \begin{bmatrix} 1 & -\cos t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 2e^t - e^t \cos t \\ e^t \end{bmatrix} \quad (3.52)$$

Although the above example illustrates the two methods of solving Eq (3.28), the real advantage in using theorem 3.5 is that it will allow one to solve higher order matrix equations when the time dependent elements can be arranged in one block. The next definition will be necessary in order to show a technique that can be used to solve a higher order matrix equation in the form of Eq (3.28).

Definition 3.1. The tensor product ( $\otimes$ ) of two matrices  $A_{m \times m}$  and  $B_{n \times n}$  is defined as

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \dots & a_{mm}B \end{bmatrix} = C_{mn \times mn} \quad (3.53)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \quad (3.54)$$

With the above definition one can readily show that by using the tensor product of matrices a one-to-one (1-1) correspondence of Eqs (3.13) and (3.31) can be written in the following form:

Eq (3.13) would transform into

$$[ (A \otimes I) - (I \otimes B^T) ] \vec{X} = \vec{C} = \begin{bmatrix} c_{11} \\ \vdots \\ c_{1n} \\ c_{21} \\ \vdots \\ c_{nn} \end{bmatrix} \quad (3.55)$$

Eq (3.31) would transform into

$$\dot{\vec{X}}(t) = [ (A \otimes I) - (I \otimes B^T) ] \vec{X}(t) - \vec{C}(t) \quad (3.56)$$

The next two examples will show how Eqs (3.55) and (3.56) would be implemented.

Example 3.4. Trying to solve Eq (3.13) of example 3.2 by Eq (3.55) would yield

$$\left[ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \right] \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \quad (3.57)$$

Multiplication of Eq (3.57) yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \quad (3.58)$$

A solution to Eq (3.58) by inspection can be readily seen as

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{---} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = X \quad (3.59)$$

The next example will use Eq (3.56) in the process of applying theorem 3.5.

Example 3.5. Solve

$$\dot{\vec{X}}(t) = \left[ \begin{array}{cc|cc} 1 & 0 & e^{-t} & e^{-t} \\ 0 & 2 & t & \cos t \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array} \right] \vec{X}(t) ; \quad \vec{X}(0) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \quad (3.60)$$

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad -C(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ t & \cos t \end{bmatrix} \quad (3.61)$$

Solving Eq (3.31) by use of Eq (3.56) yields

$$\dot{\vec{X}}(t) = \left[ \begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \vec{X}(t) + \begin{bmatrix} e^{-t} \\ e^{-t} \\ t \\ \cos t \end{bmatrix} \quad (3.62)$$

A solution to Eq (3.62) will then be of the form

$$\vec{X}(t) = e^{\hat{A}(t-t_0)} \vec{k} + \int_{t_0}^t e^{\hat{A}(t-s)} \vec{C}(s) ds \quad (3.63)$$

where

$$\hat{A} = \left[ \begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad \vec{k} = \vec{X}(t_0) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{C}(s) = \begin{bmatrix} e^{-s} \\ e^{-s} \\ s \\ \cos s \end{bmatrix} \quad (3.64)$$

Using the methods of Chapter II to compute  $e^{\hat{A}t}$ , Eq (3.63) produces

$$\overrightarrow{X(t)} = \begin{bmatrix} e^{-t} + te^{-t} \\ e^{-t} + te^{-t} \\ 2 + t^2/2 \\ 2 + \sin t \end{bmatrix} \quad (3.65)$$

With  $\overrightarrow{X(t)}$  as a solution to Eq (3.56), Eq (3.60) reduces to

$$\dot{\overrightarrow{Y(t)}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overrightarrow{Y(t)} \quad (3.66)$$

with  $\overrightarrow{Y(t_0)} = P^{-1}(t_0) \overrightarrow{X(t_0)}$ . Thus

$$\overrightarrow{Y(t_0)} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ 2 \\ 2 \end{bmatrix} \quad (3.67)$$

Thus a solution to Eq (3.66) is

$$\overrightarrow{Y(t)} = \begin{bmatrix} -3e^t \\ -7e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} \quad (3.68)$$

Therefore from the substitution  $\overrightarrow{X} = P \overrightarrow{Y}$ , the solution to the original Eq (3.60) is given by

$$\overrightarrow{X(t)} = \begin{bmatrix} 1 & 0 & (e^{-t}+te^{-t}) & (e^{-t}+te^{-t}) \\ 0 & 1 & (2 + t^2/2) & (2 + \sin t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3e^t \\ -7e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} \quad (3.69)$$



or

$$\vec{X}(t) = \begin{bmatrix} e^t + 4 t e^t \\ e^{2t} + t^2 e^{2t} + 2e^{2t} \sin t \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} \quad (3.70)$$

This example demonstrates a method that could be used to solve Eq (3.28). The evaluation of  $e^{At}$  by the method discussed in Chapter II plays an important role in this technique.

#### 3.4 Iterative Methods Used to Solve Eq (3.13)

This section will examine possible iterative techniques that could be used to solve the Liapunov equation. Continuing research on the solution of this equation has produced a varied assortment of algorithms that could be used to obtain a solution depending upon the properties of the matrices A and B of Eq (3.13). Meirovitch (Ref 12), La Salle (Ref 13), and Ma (Ref 14) give definite uses to the solution of Eq (3.13) along with possible solution techniques. Kitagawa (Ref 15) suggests an iterative technique for the solution of an equation in the form

$$X = F X F^T + S \quad (3.71)$$

where X, F, and S are n x n matrices. According to Kitagawa his algorithm works well in terms of both time and precision for high dimensional problems, and the method is also applicable to the Liapunov equation in the form

$$A^T X + XA + C = 0 \quad (3.72)$$

The iterative technique in this section was suggested by Lancaster (Ref 16) and provided a solution to Eq (3.13) when either or both A and B were singular.

Theorem 3.6. If

$$f(z) = (z + a)(z - a)^{-1} \quad (3.73)$$

where  $a \neq 0$  and  $a$  is real, and define

$$f(A) = (aI - A)^{-1}(aI + A) = U \quad (3.74)$$

$$f(B) = (aI + B)(aI - B)^{-1} = V \quad (3.75)$$

then a solution of

$$X = U X V - \frac{1}{2a} (U - I)C(V - I) \quad (3.76)$$

is also a solution to Eq (3.13).

Proof. Multiplication of Eq (3.13) by  $2a$  yields

$$2aAX + 2aXB = 2aC \quad (3.77)$$

Adding and subtracting  $a^2X$  and  $A X B$  to Eq (3.77) yields

$$a^2X + aXB + aAX + AXB - a^2X + aXB + aAX - AXB = 2aC \quad (3.78)$$

Factorization of Eq (3.78) yields

$$(aI + A)X(aI + B) - (aI - A)X(aI - B) = 2aC \quad (3.79)$$

By choosing  $a$  so that  $(aI - A)^{-1}$  and  $(aI - B)^{-1}$  exist and multiplying Eq (3.79) by these factors on the left and right gives

$$(aI-A)^{-1}(aI+A)X(aI+B)(aI-B)^{-1} - (aI-A)^{-1}(aI-A)X(aI-B)(aI-B)^{-1} =$$

$$2a(aI-A)C(aI-B)^{-1} \quad (3.80)$$

By substituting Eqs (3.74) and (3.75) into Eq (3.80) yields

$$UXV - X = 2a(aI-A)^{-1}C(aI-B)^{-1} \quad (3.81)$$

By assumption the following equations are valid

$$f(z) = \frac{(z+a)}{(z-a)} \rightarrow f(z) - 1 = \frac{z+a-z+a}{z-a} =$$

$$2a(z-a)^{-1} \rightarrow f(A) - I = 2a(A-aI)^{-1} \rightarrow \frac{1}{2a} (U-I) = (A-aI)^{-1} \quad (3.82)$$

Similarly

$$\frac{1}{2a} (V-I) = (B-aI)^{-1} \quad (3.83)$$

Thus Eq (3.81) transforms to

$$X = UXV - \frac{1}{2a} (U-I)C(V-I) \quad (3.76)$$

where

$$(A-aI)^{-1} = -(aI-A)^{-1} \quad (3.84)$$

Using the recursive algorithm yields

$$X_{n+1} = U X_n V - \frac{1}{2a} (U-I) C (V-I), \quad (n=1,2,3,\dots) \quad (3.85)$$

Lancaster did not support this iteration process with experimental data. Table I shows the results of some examples that were used to determine the usefulness of this algorithm. The iteration process con-

tinued until the answer was accurate to a specified number of significant digits. These examples were executed on a CDC 6600, which is a 60 bit machine, using single precision arithmetic. The most time consuming part of this algorithm is in determining the a parameter so that the inverse of Eqs (3.74) and (3.75) exists for a converging sequence generated by Eq (3.85). The technique found to work best was as follows:

- 1) Choose an a equal to the minimum of the elements of the matrices A, B, and C.
- 2) Assure that the inverse of Eqs (3.74) and (3.75) exists.
- 3) Determine if the sequence generated by Eq (3.85) is a converging sequence.
- 4) If the sequence is a converging one, then iterate until less than a specified tolerance.
- 5) If the sequence diverges increment a by a specified amount and begin at step 2.

The specified amount suggested in step 5 could be determined by the elements of the matrices A, B, and C. For example, if the range of the elements was from .1 to 10 then an incremental value of  $\pm 1$  was adequate. On the other hand, if the range of the values of the elements was from 1 to 100 then an incremental value of  $\pm 10$  was adequate. Different solutions could be obtained by varying the initial starting value of  $X_0$ . Also, with an initial starting value of  $X_0 = 0$  it was usually possible to obtain more than one solution by simply varying the a parameter. This technique seemed to work well for  $n \times n$  matrices where  $n < 5$ . For  $n \geq 5$  the technique is not as consistent and a good initial starting value is needed in order to assure convergence to a solution. For  $n \geq 10$  the iteration process breaks down due to the amount



of round-off error introduced. Most 10 x 10 matrices will not converge even with a good initial starting value. Example 3 of Table I is for a 10 x 10 matrix equation that could not achieve better than 2 significant digits of accuracy in spite of a good initial starting value.

Another technique suggested by Lancaster (Ref 16) determines U and V as in Eqs (3.74) and (3.75) but employs the following theorem.

Theorem 3.7. An equation of the form

$$X - U X V = C_1 \quad (3.86)$$

where

$$C_1 = \frac{1}{2a} (U - I) C (V - I) \quad (3.87)$$

has a unique solution X with series representation

$$X = \sum_{j=1}^{\infty} U^{j-1} C_1 V^{j-1} \quad (3.88)$$

if and only if  $\|U\| \cdot \|V\| < 1$ , where  $\|U\|$  and  $\|V\|$  are the norm of the matrices U and V such that the norm of the identity matrix is 1.

This algorithm proved difficult to implement since the condition  $\|U\| \cdot \|V\| < 1$  was not easily met using a variety of norms.

The next theorem suggests another algorithm that could be used to solve Eq (3.13). The notation (\*) means conjugate transpose and (+) means the generalized inverse of a matrix.

Theorem 3.8. The series generated by

$$\overline{X} = [ \sum_{k=0}^{\infty} (I - \alpha(I-G)^*(I-G))^k \alpha(I-G)^* ] (-\overline{C}) \quad (3.89)$$



where

$$0 < \alpha < \frac{2}{\|I-G\|^2} \quad (3.90)$$

is a solution to Eq (3.13) if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ is similar to } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

and

$$(I-G)(I-G)^+ \vec{C} = -\vec{C} \quad (3.91)$$

where

$$G = [I + (A \otimes I) + (I \otimes B^T)] \quad (3.92)$$

Proof. Addition and subtraction of  $X$  to Eq (3.13) yields

$$X + AXI + IXB - X = C \quad (3.93)$$

Using the tensor product of definition 3.1 for factorization yields

$$-\vec{X} - (A \otimes I)\vec{X} - (I \otimes B^T)\vec{X} + \vec{X} = -\vec{C} \quad (3.94)$$

$$\vec{X} - [I + (A \otimes I) + (I \otimes B^T)]\vec{X} = -\vec{C} \quad (3.95)$$

where  $\vec{X}$  and  $\vec{C}$  are a one-to-one correspondence of the components of the matrices  $X$  and  $C$  in vector form. By substituting Eq (3.92) into Eq (3.95) yields

$$\vec{X} - G\vec{X} = -\vec{C} \quad (3.96)$$

$$(I-G)\vec{X} = -\vec{C} \quad (3.97)$$

$$\vec{X} = (I-G)^+ (-\vec{C}) \quad (3.98)$$

Finally, by substituting the series representation for the generalized inverse as per Ben-Israel and Greville (Ref 17) one will obtain Eq (3.89).

The following example demonstrates theorem 3.8.

Example 3.5. Solve Eq (3.13) where

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (3.99)$$

Using Eq (3.92) to generate  $G$  yields

$$G = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (3.100)$$

Thus

$$(I-G) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad (3.101)$$

Using the max column norm for Eq (3.101) and using Eq (3.90) yields

$$0 < \alpha < \frac{1}{2} \quad (3.102)$$

Let  $\alpha = \frac{1}{4}$  and substituting into Eq (3.89) produces the series:

For  $k=0$

$$\vec{X} = \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \quad (3.103)$$

For  $k=1$

$$\vec{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.104)$$

Thus for the 2nd computation

$$\vec{X} = \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.105)$$

For  $k=3,4,\dots$  the results are the same as for Eq (3.105) and therefore a solution to Eq (3.13) is

$$\vec{X} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \quad (3.106)$$

This method of solving Eq (3.13) has the advantage that one does not have to compute an inverse directly before attempting to find a solution. A big disadvantage, however, is that it requires a large amount of memory to execute on a computer for  $n \times n$  matrices with  $n > 10$ .

Table I

Solution to Eq (3.13) by Iterative Method

Example 1.

Matrix A =

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Matrix B =

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Matrix C =

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$$

Matrix  $X_0$  =

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Parameter a = -2

Number of Iterations to 3 significant digits = 5

Matrix Answer X =

$$\begin{bmatrix} 1.00 & 1.00 \\ .25 & .75 \end{bmatrix}$$

Example 2.

Matrix A =

$$\begin{bmatrix} 5 & 10 & 15 \\ 7 & 14 & 21 \\ 9 & 18 & 27 \end{bmatrix}$$

Matrix B =

$$\begin{bmatrix} 4 & 8 & 12 \\ 6 & 12 & 18 \\ 3 & 6 & 9 \end{bmatrix}$$

Matrix C =

$$\begin{bmatrix} 54 & 108 & 162 \\ 78 & 156 & 234 \\ 72 & 144 & 216 \end{bmatrix}$$

Matrix  $X_0$  =

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Parameter a = -30

Number of Iterations to 3 significant digits = 5

Matrix Answer X =

$$\begin{bmatrix} .950 & 1.92 & 2.88 \\ 1.44 & 2.88 & 4.32 \\ .720 & 1.44 & 2.16 \end{bmatrix}$$

### Example 3.

Matrix A =

$$\begin{bmatrix} 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 & 45 & 50 \\ 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 & 60 \\ 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 & 36 & 40 \\ 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 & 70 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 \\ 8 & 16 & 24 & 32 & 40 & 48 & 56 & 64 & 72 & 80 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 \\ 9 & 18 & 27 & 36 & 45 & 54 & 63 & 72 & 81 & 90 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \end{bmatrix}$$



Matrix B =

1	2	4	8	16	32	64	128	256	100
0	1	2	4	8	16	32	64	128	50
0	3	6	12	24	48	96	192	175	120
100	256	128	64	32	16	8	4	2	1
50	206	73	14	21	32	77	125	85	95
7	14	21	28	35	42	49	56	63	70
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
0	10	20	30	60	70	80	100	110	150
1	8	36	57	28	56	150	90	24	71

Matrix C =

210	420	665	980	1435	2170	3465	588	10535	525
210	455	700	980	1330	1820	2590	3920	6370	3850
140	385	630	980	1540	2520	4340	7840	7385	5600
3745	9450	5215	3220	2345	2030	1995	2100	2275	2485
1855	7420	2870	910	1260	1750	3430	5215	3920	4375
525	1050	1575	2100	2625	3150	3675	4200	4725	5250
385	770	1155	1540	1925	2310	2695	3080	3465	3850
665	1330	1995	2660	3325	3990	4655	5320	5985	6650
35	420	805	1190	2275	2660	3045	3780	4165	5600
385	980	2310	3395	2730	4060	7700	5950	3990	5985

Matrix  $X_0$  =

34	0	0	0	0	0	0	0	0	0
0	34	0	0	0	0	0	0	0	0
0	0	34	0	0	0	0	0	0	0
0	0	0	34	0	0	0	0	0	0
0	0	0	0	34	0	0	0	0	0
0	0	0	0	0	34	0	0	0	0
0	0	0	0	0	0	34	0	0	0
0	0	0	0	0	0	0	34	0	0
0	0	0	0	0	0	0	0	34	0
0	0	0	0	0	0	0	0	0	34

Parameter a = -900

Number of Iterations to 2 significant digits = 4

Matrix Answer X =

$$\begin{bmatrix} 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \end{bmatrix}$$

#### IV. Reducibility Using Solutions of the Riccati Equation

This chapter will extend the ideas of Chapter III to more general block forms of the equation

$$\dot{\vec{X}}(t) = \tilde{A}(t) \vec{X}(t) \quad (1.1)$$

The first section of this chapter will apply transformations to Eq (1.1) for the constant case by solving the Riccati equation associated with Eq (1.1). In the second section the discussion will cover specific time dependent cases, and the final section will present some methods that could be used to solve the Riccati equation.

##### 4.1 Reducibility of Eq (1.1) for the Constant Case

Theorem 4.1. The differential system of Eq (1.1) in the form

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c} -B & D \\ \hline -C & A \end{array} \right] \vec{X}(t) \quad (4.1)$$

where A,B,C and D are  $n \times n$  matrices belonging to the field of real numbers, is kinematically similar to

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{c|c} XD+A & 0 \\ \hline D & -DX-B \end{array} \right] \vec{Y}(t) \quad (4.2)$$

by the transformation

$$\vec{X}(t) = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X \end{array} \right] \vec{Y}(t) \quad (4.3)$$

where X is a solution to the Riccati matrix equation

$$AX + XB + C + XDX = 0 \quad (4.4)$$

Proof. Let

$$\vec{X}(t) = P \vec{Y}(t) \quad (4.5)$$

where

$$P = \left[ \begin{array}{c|c} 0 & I \\ I & -X \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} X & I \\ I & 0 \end{array} \right] \quad (4.6)$$

Differentiating Eq (4.5) yields

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c} 0 & I \\ I & -X \end{array} \right] \dot{\vec{Y}}(t) \quad (4.7)$$

Substitution of Eq (4.5) into Eq (4.1) and equating with Eq (4.7) yields

$$\left[ \begin{array}{c|c} -B & D \\ -C & A \end{array} \right] \left[ \begin{array}{c|c} 0 & I \\ I & -X \end{array} \right] \vec{Y}(t) = \left[ \begin{array}{c|c} 0 & I \\ I & -X \end{array} \right] \dot{\vec{Y}}(t) \quad (4.8)$$

Multiplication of Eq (4.8) on the left by  $P^{-1}$  yields

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{c|c} XD+A & -AX-XB-C-XDX \\ D & -DX-B \end{array} \right] \vec{Y}(t) \quad (4.9)$$

which reduces to Eq (4.2) with  $X$  a solution to Eq (4.4).

Example 4.1. Reduce the differential system

$$\dot{\vec{X}}(t) = \left[ \begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ -4 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 \end{array} \right] \vec{X}(t) \quad (4.10)$$

Let

$$-B = \left[ \begin{array}{cc} 0 & -1 \\ 0 & 2 \end{array} \right], \quad D = \left[ \begin{array}{cc} 0 & 0 \\ 1 & -1 \end{array} \right], \quad -C = \left[ \begin{array}{cc} -4 & 0 \\ 0 & -1 \end{array} \right],$$

$$A = \left[ \begin{array}{cc} 0 & 0 \\ 1 & -2 \end{array} \right] \quad (4.11)$$

A solution to Eq (4.4) is

$$X = \left[ \begin{array}{cc} 0 & -1 \\ -4 & -2 \end{array} \right] \quad (4.12)$$

Thus using Eq (4.5) with

$$P = \left[ \begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 2 \end{array} \right], \quad P^{-1} = \left[ \begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ -4 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad (4.13)$$

the substitution into Eq (4.10) will by theorem 4.1 reduce the system to

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -4 & 0 \end{array} \right] \vec{Y}(t) \quad (4.14)$$

The following theorems show the flexibility of block reducing Eq (1.1) by using Eq (4.4). Depending upon the original differential system, a proper choice of A,B,C, and D as sub-matrices may allow the solution X in Eq (4.4) to be more easily solved. The other alternative is to vary the P matrix in Eq (4.5) so that the resulting reduced differential system is in a simpler form.

Theorem 4.2. The differential system of Eq (1.1) in the form

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c} A & C \\ \hline -D & -B \end{array} \right] \vec{X}(t) \quad (4.15)$$

where A,B,C,D are  $n \times n$  matrices belonging to the field of real numbers, is kinematically similar to

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{c|c} -DX-B & -D \\ \hline 0 & A+XD \end{array} \right] \vec{Y}(t) \quad (4.16)$$

by the transformation

$$\vec{X}(t) = \left[ \begin{array}{c|c} X & I \\ \hline I & 0 \end{array} \right] \vec{Y}(t) \quad (4.17)$$

where X is a solution to Eq (4.4).



Proof. As in theorem 4.1, let

$$P = \left[ \begin{array}{c|c} X & I \\ \hline I & 0 \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X \end{array} \right] \quad (4.18)$$

then by differentiating Eq (4.17) and substitution into Eq (4.15) yields

$$\left[ \begin{array}{c|c} A & C \\ \hline -D & -B \end{array} \right] \left[ \begin{array}{c|c} X & I \\ \hline I & 0 \end{array} \right] \overrightarrow{Y(t)} = \left[ \begin{array}{c|c} X & I \\ \hline I & 0 \end{array} \right] \overrightarrow{\dot{Y}(t)} \quad (4.19)$$

Multiplication of Eq (4.19) on the left by  $P^{-1}$  reduces to Eq (4.16)

with  $X$  a solution to Eq (4.4).

Theorem 4.3. The differential system in the form of Eq (4.15) is kinematically similar to

$$\overrightarrow{\dot{Y}(t)} = \left[ \begin{array}{c|c} A+XD & 0 \\ \hline -D & -DX-B \end{array} \right] \overrightarrow{Y(t)} \quad (4.2)$$

by the transformation

$$\overrightarrow{X(t)} = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \overrightarrow{Y(t)} \quad (4.20)$$

where  $X$  is a solution to Eq (4.4).

Proof. Let

$$P = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c} I & -X \\ \hline 0 & I \end{array} \right] \quad (4.21)$$

then by differentiating Eq (4.20) and substitution into Eq (4.15) yields

$$\left[ \begin{array}{c|c} A & C \\ \hline -D & -B \end{array} \right] \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \overrightarrow{Y(t)} = \left[ \begin{array}{c|c} I & X \\ \hline 0 & I \end{array} \right] \overrightarrow{\dot{Y}(t)} \quad (4.22)$$

Multiplication of Eq (4.22) on the left by  $P^{-1}$  reduces to Eq (4.2) with  $X$  a solution to Eq (4.4).

Example 4.2. Reduce by theorem 4.3 the differential system

$$\dot{\vec{x}}(t) = \left[ \begin{array}{cc|cc} 3 & 0 & 4 & 3 \\ 1 & -2 & 4 & 1 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 \end{array} \right] \vec{x}(t) \quad (4.23)$$

Let

$$A = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}, \quad -B = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 3 \\ 4 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.24)$$

A solution to Eq (4.4) is

$$X = \begin{bmatrix} 0 & -1 \\ -4 & -2 \end{bmatrix} \quad (4.25)$$

Using Eq (4.21)

$$P = \left[ \begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & -4 & -2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right], \quad P^{-1} = \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (4.26)$$

Eq (4.23) by theorem 4.3 reduces to

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{cc|cc} 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 \\ 0 & 1 & -4 & 0 \end{array} \right] \vec{Y}(t) \quad (4.27)$$

The following theorems expand the ideas already presented to other forms of Eq (1.1) by using other partitioning schemes. As in the other cases there is flexibility in what one calls the matrices A,B,C, and D and also in the choice of the P matrix.

Theorem 4.4. The differential system of Eq (1.1) in the form

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c|c} -A & 0 & C \\ \hline 0 & I & 0 \\ \hline -D & 0 & B \end{array} \right] \vec{X}(t) \quad (4.28)$$

is kinematically similar to

$$\dot{\vec{Y}}(t) = \left[ \begin{array}{c|c|c} -A-XD & 0 & 0 \\ \hline 0 & I & 0 \\ \hline -D & 0 & DX+B \end{array} \right] \vec{Y}(t) \quad (4.29)$$

where A, B, C, and D are  $n \times n$  matrices belonging to the field of real numbers by the transformation

$$\vec{X}(t) = \left[ \begin{array}{c|c|c} I & 0 & -X \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \vec{Y}(t) \quad (4.30)$$

where X is a solution to Eq (4.4).

Proof. Let

$$P = \left[ \begin{array}{c|c|c} I & 0 & -X \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c|c} I & 0 & X \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \quad (4.31)$$

then by differentiating Eq (4.30) and substitution into Eq (4.28) yields

$$\left[ \begin{array}{c|c|c} -A & 0 & C \\ \hline 0 & I & 0 \\ \hline -D & 0 & B \end{array} \right] \left[ \begin{array}{c|c|c} I & 0 & -X \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \dot{\vec{Y}}(t) = \left[ \begin{array}{c|c|c} I & 0 & -X \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \dot{\vec{Y}}(t) \quad (4.32)$$

Multiplication of Eq (4.32) on the left by  $P^{-1}$  reduces Eq (4.29) with X a solution to Eq (4.4).

Theorem 4.5. The differential system of Eq (1.1) in the form

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c|c} B & 0 & -D \\ \hline 0 & I & 0 \\ \hline C & 0 & -A \end{array} \right] \vec{X}(t) \quad (4.33)$$

is kinematically similar to

$$\dot{\overrightarrow{Y}}(t) = \left[ \begin{array}{c|c|c} B+DX & 0 & -D \\ \hline 0 & I & 0 \\ \hline 0 & 0 & -XD-A \end{array} \right] \overrightarrow{Y}(t) \quad (4.34)$$

by the transformation

$$\overrightarrow{X}(t) = \left[ \begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline -X & 0 & I \end{array} \right] \overrightarrow{Y}(t) \quad (4.35)$$

where  $X$  is a solution to Eq (4.4).

Proof. The proof is similar to theorem 4.4 with

$$P = \left[ \begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline -X & 0 & I \end{array} \right], \quad P^{-1} = \left[ \begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline X & 0 & I \end{array} \right] \quad (4.36)$$

Example 4.3. Reduce the differential system

$$\dot{\overrightarrow{X}}(t) = \left[ \begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 4 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{array} \right] \overrightarrow{X}(t) \quad (4.37)$$

Using theorem 4.4, let

$$\begin{aligned} -A &= \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \\ -D &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (4.38)$$

A solution to Eq (4.4) is

$$X = \begin{bmatrix} 0 & -1 \\ -4 & -2 \end{bmatrix} \quad (4.39)$$

and thus by substitution for  $X$  into Eq (4.29), Eq (4.37) reduces to



$$\dot{\vec{Y}}(t) = \left[ \begin{array}{cc|cc|cc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 4 & 0 \end{array} \right] \vec{Y}(t) \quad (4.40)$$

#### 4.2 Reducibility of Time Dependent Forms of Eq (1.1)

This section will show how the same basic techniques already used for constant matrices is also applicable to matrices with variable elements. Although various choices of the sub-matrices A, B, C, and D is possible by choosing the proper transformation P(t) matrix, the form of Eq (1.1) to be examined in this section is

$$\dot{\vec{X}}(t) = \left[ \begin{array}{c|c} -B(t) & D(t) \\ \hline -C(t) & A(t) \end{array} \right] \vec{X}(t) \quad (4.41)$$

where A, B, C, and D are n x n matrices with polynomial elements and real coefficients such that

$$A(t) = A_0 + A_1 t + A_2 t^2 + \dots + A_p t^p \quad (4.42)$$

$$B(t) = B_0 + B_1 t + B_2 t^2 + \dots + B_q t^q \quad (4.43)$$

$$C(t) = C_0 + C_1 t + C_2 t^2 + \dots + C_r t^r \quad (4.44)$$

$$D(t) = D_0 + D_1 t + D_2 t^2 + \dots + D_s t^s \quad (4.45)$$

Theorem 4.6. Let A, B, C, and D be n x n matrices with polynomial elements as defined in Eqs (4.42), (4.43), (4.44), and (4.45) and let  $X(t_0) = X_0$ . Then a power series solution to

$$\dot{X}(t) = A(t)X(t) + X(t)B(t) + C(t) + X(t)D(t)X(t) \quad (4.46)$$



is

$$X(t) = X_0 + X_1 t + X_2 t^2 + \dots + X_n t^n \quad (4.47)$$

Proof. Substitution of Eq (4.47) into Eq (4.46) yields

$$\begin{aligned} & [A_0 + A_1 t + A_2 t^2 + \dots + A_p t^p] [X_0 + X_1 t + \dots + X_n t^n] + [X_0 + X_1 t + \dots + X_n t^n] \\ & [B_0 + B_1 t + \dots + B_q t^q] + [C_0 + C_1 t + \dots + C_r t^r] + [X_0 + X_1 t + \dots + X_n t^n] \\ & [D_0 + D_1 t + \dots + D_s t^s] [X_0 + X_1 t + \dots + X_n t^n] = [X_1 + 2X_2 t + 3X_3 t^2 + \dots + nX_n t^{n-1}] \end{aligned} \quad (4.48)$$

Multiplication of Eq (4.48) and matching powers of  $t$  yields the following set of equations

$$A_0 X_0 + X_0 B_0 + C_0 + X_0 D_0 X_0 = X_1 \quad (4.49)$$

$$A_0 X_1 + A_1 X_0 + X_0 B_1 + X_1 B_0 + C_1 + X_0 D_0 X_1 + X_0 D_1 X_0 + X_1 D_0 X_0 = 2X_2 \quad (4.50)$$

$$\begin{aligned} & A_0 X_2 + A_1 X_1 + A_2 X_0 + X_0 B_2 + X_1 B_1 + X_2 B_0 + C_2 + X_0 D_0 X_2 + X_0 D_1 X_1 + X_0 D_2 X_0 + \\ & X_1 D_0 X_1 + X_1 D_1 X_0 + X_2 D_0 X_0 = 3X_3 \end{aligned} \quad (4.51)$$

-----

The matching process would then continue until  $X_n$  was matched.

Theorem 4.6 can be extended under certain conditions to include the case where  $A(t)$ ,  $B(t)$ ,  $C(t)$  and  $D(t)$  have elements as functions which can be approximated by a polynomial.

Theorem 4.7. The differential system of Eq (4.41) is kinematically similar to

$$\dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} \frac{X(t)D(t)+A(t)}{D(t)} & 0 \\ \hline -D(t)X(t)-B(t) \end{array} \right] \overline{Y}(t) \quad (4.52)$$

by the transformation

$$\overline{X}(t) = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right] \overline{Y}(t) \quad (4.53)$$

where  $X(t)$  in Eq (4.47) is a solution to Eq (4.46).

Proof. Let

$$P = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right], \quad P^{-1}(t) = \left[ \begin{array}{c|c} X(t) & I \\ \hline I & 0 \end{array} \right] \quad (4.54)$$

Differentiating Eq (4.53) yields

$$\dot{\overline{X}}(t) = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right] \dot{\overline{Y}}(t) + \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & -\dot{X}(t) \end{array} \right] \overline{Y}(t) \quad (4.55)$$

Equating Eq (4.41) with Eq (4.55), and substitution of Eq (4.53) into Eq (4.41) yields

$$\left[ \begin{array}{c|c} -B(t) & D(t) \\ \hline -C(t) & A(t) \end{array} \right] \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right] \overline{Y}(t) = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right] \dot{\overline{Y}}(t) + \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & -\dot{X}(t) \end{array} \right] \overline{Y}(t) \quad (4.56)$$

Multiplication and then the addition of factors of  $\overline{Y}(t)$  yields

$$\left[ \begin{array}{c|c} D(t) & -B(t)-D(t)X(t) \\ \hline A(t) & -C(t)-A(t)X(t)+\dot{X}(t) \end{array} \right] \overline{Y}(t) = \left[ \begin{array}{c|c} 0 & I \\ \hline I & -X(t) \end{array} \right] \dot{\overline{Y}}(t) \quad (4.57)$$

Multiplication on the left by  $P^{-1}(t)$  yields

$$\dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} \frac{X(t)D(t)+A(t)}{D(t)} & \frac{-X(t)B(t)-X(t)D(t)X(t)-C(t)-A(t)X(t)+\dot{X}(t)}{-D(t)X(t)-B(t)} \\ \hline -D(t)X(t)-B(t) \end{array} \right] \overline{Y}(t) \quad (4.58)$$

which reduces to Eq (4.52) with  $X(t)$  a solution to Eq (4.46).

The next example will apply theorems 4.6 and 4.7.

Example 4.4. Reduce

$$\dot{\vec{X}}(t) = \left[ \begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline (t^2-t-1) & (2t-3) & t & (t+1) \\ (-2t^2+2t+1) & (-3t+4) & 2 & t \end{array} \right] \vec{X}(t) \quad (4.59)$$

with

$$\vec{X}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad (4.60)$$

Let

$$A = \begin{bmatrix} t & (t+1) \\ 2 & t \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} (-t^2+t+1) & (-2t+3) \\ (2t^2-2t-1) & (3t-4) \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4.61)$$

Choose

$$X(t_0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = X_0 \quad (4.62)$$

Applying Eq (4.49) for a solution to Eq (4.46) yields

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} =$$

$$X_1 \quad (4.63)$$

$$X_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \quad (4.64)$$

Similarly applying Eqs (4.50) and (4.51) yields

$$x_2 = x_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (4.65)$$

Thus using Eq (4.47) a solution to Eq (4.46) is

$$x(t) = \begin{bmatrix} t & 1 \\ -t & -1 \end{bmatrix} \quad (4.66)$$

Therefore by theorem 4.7 with

$$P(t) = \left[ \begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -t & -1 \\ 0 & 1 & t & 1 \end{array} \right] \quad (4.67)$$

Eq (4.59) reduces to

$$\dot{\overline{Y}}(t) = \left[ \begin{array}{cc|cc} 2t & (t+2) & 0 & 0 \\ (2-t)(t-1) & & 0 & 0 \\ 1 & 0 & (-t-1) & -2 \\ 0 & 1 & t & 2 \end{array} \right] \overline{Y}(t) \quad (4.68)$$

with  $\overline{Y}(t_0) = P^{-1}(t_0)X(t_0)$ , so that

$$\overline{Y}(t_0) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (4.69)$$

The differential system of Eq (4.68) can now be reduced still further by solving Eq (4.46) with  $D(t) = 0$  and using the following theorem.

Theorem 4.8. The differential system in the form

$$\dot{\overline{X}}(t) = \left[ \begin{array}{c|c} -B(t) & 0 \\ -C(t) & A(t) \end{array} \right] \overline{X}(t) \quad (4.70)$$

where A, B, and C are defined in Eqs (4.42), (4.43), and (4.44), is kinematically similar to



$$\dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} -B(t) & 0 \\ \hline 0 & A(t) \end{array} \right] \overline{Y}(t) \quad (4.71)$$

with the transformation

$$\overline{X}(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline -X(t) & I \end{array} \right] \overline{Y}(t) \quad (4.72)$$

where  $X(t)$  is a solution Eq (4.46) with  $D(t) = 0$ .

Proof. Let

$$P(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline -X(t) & I \end{array} \right], \quad P^{-1}(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline X(t) & I \end{array} \right] \quad (4.73)$$

As in theorem 4.7, differentiating Eq (4.72), equating with Eq (4.70) with the substitution of Eq (4.72) yields

$$\left[ \begin{array}{c|c} -B(t) & 0 \\ \hline -C(t) & A(t) \end{array} \right] \left[ \begin{array}{c|c} I & 0 \\ \hline -X(t) & I \end{array} \right] \overline{Y}(t) = \left[ \begin{array}{c|c} I & 0 \\ \hline -X(t) & I \end{array} \right] \dot{\overline{Y}}(t) + \left[ \begin{array}{c|c} 0 & 0 \\ \hline -X(t) & 0 \end{array} \right] \overline{Y}(t) \quad (4.74)$$

Then multiplication on the left by  $P^{-1}(t)$  and grouping the factors of  $\overline{Y}(t)$  yields

$$\dot{\overline{Y}}(t) = \left[ \begin{array}{c|c} -B(t) & 0 \\ \hline -A(t)X(t) - X(t)B(t) - C(t) + X(t) & A(t) \end{array} \right] \overline{Y}(t) \quad (4.75)$$

which reduces to Eq (4.71) with  $X(t)$  a solution to Eq (4.46) with  $D(t) = 0$ .

One can readily see that the process could be carried still further by separating the system of Eq (4.71) into two separate systems and then alternating theorems 4.7 and 4.8 until a total reduction was obtained. The next section will demonstrate this technique for the constant case.



### 4.3 Solutions to Eq (4.4)

This section will examine solution techniques to solve the Riccati equation. As with the Liapunov equation, new methods to solve various forms of Eq (4.4) are under constant study. The Riccati equation by itself plays an important part in the optimal estimation and filtering problem pursued by Kalman (Ref 18) and also in orbital trajectory optimization as investigated by Bryson and Ho (Ref 19), Breakwell (Ref 20), and McReynolds (Ref 21). As was mentioned in Chapter I, a thesis by Leuthauser (Ref 4) investigates ways to obtain solutions to Eq (4.4), and this section will apply some of the methods suggested by Leuthauser.

Probably one of the best existence theorems in obtaining a solution to Eq (4.4) was done by J. Jones (Ref 22), and it has a direct application to the theorems of this chapter.

Theorem 4.9. (Jones (Ref 22)). Let  $f_Y(\lambda)$  be any polynomial of degree  $n > 1$  in  $\lambda$  with coefficients belonging to the field of real numbers such that

$$R = \left[ \begin{array}{c|c} -B & D \\ \hline -C & A \end{array} \right], \quad f_Y(R) = \left[ \begin{array}{c|c} U & M \\ \hline V & N \end{array} \right] \quad (4.80)$$

where A, B, C, and D are of order  $n \times n$ ,  
then a solution of

$$(X, I) f_Y(R) = (0, 0) \quad (4.81)$$

with  $U^{-1}$  or  $M^{-1}$  existing, or a solution of

$$f_Y(R) \left[ \begin{array}{c} I \\ \hline -X \end{array} \right] = \left[ \begin{array}{c} 0 \\ \hline 0 \end{array} \right] \quad (4.82)$$

with  $M^{-1}$  or  $N^{-1}$  existing, is also a solution Eq (4.4).

Theorem 4.9 when combined with theorem 4.1 can constitute one theorem as was originally presented by Jones, Lukes, and Dolan (Ref 23). The next example will use theorem 4.1 to reduce a differential system and also theorem 4.9 along with the work done by Leuthauser to solve Eq (4.4).

Example 4.5. Reduce the differential system

$$\vec{X}(t) = \left[ \begin{array}{cc|cc} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ \hline -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{array} \right] \vec{X}(t) \quad (4.83)$$

Substitution of R for the matrix of Eq (4.83) and finding the characteristics roots yields

$$|\lambda I - R| = \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = 0$$

$$\lambda = 1, 1, 1, 2 \quad (4.84)$$

Constructing the table of characteristic roots as suggested by Leuthauser yields

TABLE II

Combinations of Characteristic Roots of R

	1	1	1	2
1	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)(\lambda-2)$
1	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)(\lambda-2)$
1	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)^2$	$(\lambda-1)(\lambda-2)$
2	$(\lambda-1)(\lambda-2)$	$(\lambda-1)(\lambda-2)$	$(\lambda-1)(\lambda-2)$	$(\lambda-2)^2$

Table II shows that there are only 3 possibilities to check for a solution to Eq (4.4):  $(\lambda-1)^2$ ,  $(\lambda-1)(\lambda-2)$ , and  $(\lambda-2)^2$ . Choosing  $(\lambda-1)^2$  as a possible solution

$$f_{\lambda}(R) = (R-I)^2 = R^2 - 2R + I \quad (4.85)$$

Substituting the matrix R into Eq (4.85) and using Eq (4.80) yields

$$\left[ \begin{array}{c|c} U & M \\ \hline V & N \end{array} \right] = \left[ \begin{array}{cc|cc} -3 & -13 & -13 & -7 \\ -3 & -18 & -18 & -9 \\ \hline 2 & 12 & 12 & 6 \\ 4 & 19 & 19 & 10 \end{array} \right] \quad (4.86)$$

Solving the equations

$$XU + V = 0 \quad (4.87)$$

$$XM + N = 0 \quad (4.88)$$

from Eq (4.81) yields from solving Eq (4.87)

$$X = \begin{bmatrix} 0 & .668 \\ 1 & .336 \end{bmatrix} \quad (4.89)$$

A check of this solution with Eq (4.88) confirms by theorem 4.9 that X is a solution to Eq (4.4). Thus with one valid solution X, by theorem 4.1 the system of Eq (4.83) reduces to

$$\dot{\overrightarrow{Y}}(t) = \left[ \begin{array}{cc|cc} 1.34 & .33 & 0 & 0 \\ -.32 & .66 & 0 & 0 \\ \hline -1 & -4 & 3 & 2 \\ 5 & -4 & 6 & 2 \end{array} \right] \overrightarrow{Y}(t) \quad (4.90)$$

In both Chapters III and IV, the kinematic similarity theorems have been limited to the case where A, B, C, and D were n x n matrices. All these theorems are, however, just as valid for the partitioning of the sub-matrices A, B, C, and D into rectangular blocks. The only real requirement is that the resulting Liapunov or Riccati equation is conformable to obtaining a solution. Although theorem 4.9 as originally done by

J. Jones was specifically for the case where the sub-matrices were of order  $n \times n$ , the theorem can be extended to cover rectangular matrices.

Theorem 4.10. Let  $f_Y(\lambda)$  be any polynomial of degree  $n > 1$  in  $\lambda$  with coefficients belonging to the field of real numbers such that  $R, f_Y(R)$  as given by Eq (4.80) for  $A_{MXM}, B_{PXP}, C_{MXP}, D_{PXM}$ . Then a solution of Eq (4.81) with  $U^{-1}$  existing or a solution to Eq (4.82) with  $N^{-1}$  existing for  $X_{MXP}$  is also a solution to Eq (4.4).

Proof. Similar to the proof done by J. Jones (Ref 22) one must first show that if a solution of Eq (4.81) exists with  $U^{-1}$  existing then a solution is also a solution of Eq (4.4). Let such a solution  $X$  exist. Now the matrices  $R, f_Y(R)$  commute so we have the following identities:

$$DV + UB = BU - MC; \quad UD + MA = DN - BM \quad (4.91)$$

$$AV + VB = CU - NC; \quad CM + VD = -NA + AN$$

Making use of Eqs (4.91) and the above solution  $X$  yields

$$\begin{aligned} 0 &= -(XM + N)C \\ 0 &= -XUB - XMC + XUB - NC \\ 0 &= -X(UB + MC) + XUB - NC \\ 0 &= -X(BU - DV) + XUB - NC \\ 0 &= -XBU - XDXU + XUB - NC \\ 0 &= -XDXU - AXU - XBU + AXU + XUB - NC \\ 0 &= -XDXU - AXU - XBU - AV - VB - NC \\ 0 &= -XDXU - XBU - AXU - CU \\ 0 &= -(XDX + AX + XB + C)U \end{aligned} \quad (4.92)$$

and since  $U^{-1}$  exists  $X$  is a solution to Eq (4.4).

Next let  $X$  be a solution of Eq (4.82) with  $N^{-1}$  existing. Making use of Eqs (4.91) yields



$$\begin{aligned}
0 &= -A(V-NX) \\
0 &= (-CU+NC+VB)+(CM+VD+NA)X \\
0 &= -CMX+NC+VB+CMX+NXDX+NAX \\
0 &= -AV+CU+NXDX+NAX \\
0 &= VB+NC+NXDX+NAX \\
0 &= NXB+NC+NXDX+NAX \\
0 &= N(XDX+AX+XB+C)
\end{aligned}
\tag{4.93}$$

Thus Eqs (4.93) imply that  $X$  is a solution to Eq (4.4).

The next example will demonstrate how the reducibility theorems can implement a rectangular partitioning scheme, and it will also show how the alternate use of theorems 4.1 and theorem 3.4 can reduce a differential system like Eq (1.1) to diagonalized systems.

Example 4.6. Reduce

$$\dot{\vec{X}}(t) = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{bmatrix} \vec{X}(t)
\tag{4.94}$$

Using theorem 4.1 with

$$-B=5, \quad -C=\begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad D=(2,2), \quad A=\begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}
\tag{4.95}$$

A solution to Eq (4.4) is

$$X = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}
\tag{4.96}$$

Thus Eq (4.94) reduces to

$$\dot{\vec{Y}}(t) = \begin{bmatrix} 2 & -4 & 0 \\ -5 & 1 & 0 \\ 2 & 2 & 6 \end{bmatrix} \vec{Y}(t)
\tag{4.97}$$

Now using theorem 3.4 with



$$B = \begin{bmatrix} 2 & -4 \\ -5 & 1 \end{bmatrix}, \quad -C = (2, 2), \quad A = 6 \quad (4.98)$$

A solution to Eq (3.13) is

$$X = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix} \quad (4.99)$$

Therefore Eq (4.97) reduces to

$$\dot{\vec{z}}(t) = \begin{bmatrix} 2 & -4 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \vec{z}(t) \quad (4.100)$$

Eq (4.100) can now be broken into two separate systems

$$\dot{\vec{z}}_{1,2}(t) = \begin{bmatrix} 2 & -4 \\ -5 & 1 \end{bmatrix} \vec{z}_{1,2}(t) \quad (4.101)$$

and

$$\dot{\vec{z}}_3(t) = 6 \vec{z}_3(t) \quad (4.102)$$

Applying theorem 4.1 on Eq (4.101), let

$$-B=2, \quad -C=-5, \quad D=-4, \quad A=1 \quad (4.103)$$

A solution to Eq (4.4) is

$$X = 1 \quad (4.104)$$

Thus Eq (4.101) reduces to

$$\dot{\vec{R}}(t) = \begin{bmatrix} -3 & 0 \\ -4 & 6 \end{bmatrix} \vec{R}(t) \quad (4.105)$$

Using theorem 3.4 again yields

$$B=-3, -C=-4, A=6 \quad (4.106)$$

Solving Eq (3.13)

$$X = 4/9 \quad (4.107)$$

Therefore Eq (4.105) reduces to

$$\overline{S}(t) = \begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix} \overline{S}(t) \quad (4.108)$$

Thus by saving the solutions to Eqs (4.4) and (3.13), a solution to the original differential system of Eq (4.94) can be found by solving the diagonalized systems of Eqs (4.108) and (4.102).

By examining the matrix used in this example, which was found in Nerring (Ref 7:216), the classical reduction technique for reducing this system would require finding the generalized eigenvectors in order to reduce the matrix to a Jordan form.

Iterative Methods to Solve Eq (4.4). The iterative methods investigated in this section follow from the work done by Leuthauser (Ref 4), and it is an extension of Lancaster's method (Ref 16) applied to Eq (4.4).

Theorem 4.11 (Leuthauser (Ref 4)). If

$$f(z) = (z+a)(z-a)^{-1} \quad (4.109)$$

where  $a \neq 0$  and  $a$  is real.

and if  $U$  and  $V$  are as defined in theorem 3.6 then a solution of

$$X = UXV - \frac{1}{2a} (U-I)C(V-I) - \frac{1}{2a} (U-I)XDX(V-I) \quad (4.110)$$

is also a solution to Eq (4.4).

The process used to determine the  $a$  parameter was identical to the one discussed in Chapter III in finding a solution to the Liapunov equation. Different solutions could be obtained by varying the initial starting value  $X_0$ . Also, with an initial starting value  $X_0 = 0$  it was usually possible to obtain more than one solution by simply varying the  $a$  parameter. Table III shows the results of some examples that were executed using single precision arithmetic on the CDC 6600. As with the Liapunov equation for  $n \times n$  matrices where  $n < 5$ , solutions were readily obtained. However, for  $n \geq 5$  a good initial starting value was needed. For  $n \geq 10$  it is difficult to make the iterative process converge even with a good initial approximation. Example 3 of Table III is for  $10 \times 10$  matrices which could only be made accurate to 2 significant digits before iterating away from a solution. Of special note in Table III is Example 2 which would not yield a solution by the method of Example 4.5 because matrices  $U, M$ , and  $N$  were all singular.

This iterative technique can also find the square root of a matrix by setting  $A=B=0$ ,  $D = -I$ , and  $C$  as the input matrix. Table IV shows the results of a square root example. For the square root case once an  $a$  parameter was found,  $-a$  would yield the opposite sign solution. The initial starting value of  $X_0 = 0$  seemed to work well for  $n \leq 10$ .

Another iterative technique investigated was an adaptation of a method suggested by Carnhahan (Ref 24) for solutions of equations in the form

$$X = F(X) \tag{4.111}$$

By letting  $F(X)$  be Eq (4.110) then similar to Carnhahan a new iterative scheme in the form

$$X = (I-K)X + K F(X) \quad (4.112)$$

where  $K$  is a matrix parameter that speeds convergence, could be used to solve Eq (4.4).

The same examples of Table III were also tested on Eq (4.112) with very little difference in the results except for example 2 which reached an answer in 20 iterations.



TABLE III

Solutions to Eq (4.4) by Iterative MethodExample 1.

$$\text{Matrix A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Matrix B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{Matrix C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{Matrix D} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Matrix } X_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Parameter  $a = 3$

Number of Iterations to 3 significant digits = 4

$$\text{Matrix Answer X} = \begin{bmatrix} .414 & -.414 \\ -.414 & .414 \end{bmatrix}$$

Parameter  $a = -7$

Number of Iterations to 3 significant digits = 8

$$\text{Matrix Answer X} = \begin{bmatrix} -2.41 & 2.41 \\ 2.41 & -2.41 \end{bmatrix}$$

Example 2.

$$\text{Matrix A} = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\text{Matrix B} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -4 & 0 \\ 1 & -1 & -2 \end{bmatrix}$$



$$\text{Matrix C} = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\text{Matrix D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\text{Matrix } X_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Parameter  $a = -4$

Number of iterations to 3 significant digits = 31

$$\text{Matrix Answer} = \begin{bmatrix} -.939 & .939 & -.939 \\ -.939 & .939 & -.939 \\ -.939 & .939 & -.939 \end{bmatrix}$$

### Example 3.

Matrix A =

5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
4	8	12	16	20	24	28	32	36	40
7	14	21	28	35	42	49	56	63	70
3	6	9	12	15	18	21	24	27	30
8	16	24	32	40	48	56	64	72	80
2	4	6	8	10	12	14	16	18	20
9	18	27	36	45	54	63	72	81	90
1	2	3	4	5	6	7	8	9	10
10	20	30	40	50	60	70	80	90	100

Matrix B =

1	2	4	8	16	32	64	128	256	100
0	1	2	4	8	16	32	64	128	50
0	3	6	12	24	48	96	192	175	120
100	256	128	64	32	16	8	4	2	1
50	206	73	14	21	32	77	125	85	95
7	14	21	28	35	42	49	56	63	70
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100
0	10	20	30	60	70	80	100	110	150
1	8	36	57	28	56	150	90	24	71

Matrix C =

-	714	- 9380	-21385	-28420	-35315	-41930	-47985	-52920	-55615	3500
-	3465	-10570	-14000	-17395	-43995	-181930	-70910	-106330	-18130	1750
-	4760	-13090	-58170	-72520	-120960	-107730	-69160	-41160	-11715	4200
-	118755	-64050	-43785	-21280	-89530	-34720	-22505	-181650	-22225	35
-	102270	-66080	-43680	-109340	-53865	-71750	-82320	-20510	-58555	3325
-	43575	-50400	-29050	-34650	-46375	-70350	-20825	-106050	-93275	2450
-	21665	-24955	-42945	-86660	1925	2310	-46305	-143920	-70035	3150
-	560	- 1120	- 1680	- 2240	-2800	-3360	4655	- 4480	- 5040	3500
-	61215	-24080	-244195	-151935	-83475	-70840	-45955	-118720	-32585	5250
	350	700	1050	1400	1750	2100	2450	2800	3150	0

Matrix D =

6	8	18	24	30	36	42	48	54	60
3	9	12	15	37	150	60	90	20	37
4	11	48	60	100	90	60	40	20	70
100	60	40	20	75	30	20	150	20	0
85	60	38	90	45	60	70	21	51	77
36	42	25	30	40	60	20	90	80	31
18	21	36	72	0	0	40	120	60	200
1	2	3	4	5	6	0	8	9	10
50	20	200	125	70	60	40	100	30	0
2	4	6	8	20	40	90	15	7	6

Matrix  $X_0$  =

34	0	0	0	0	0	0	0	0	0
0	34	0	0	0	0	0	0	0	0
0	0	34	0	0	0	0	0	0	0
0	0	0	34	0	0	0	0	0	0
0	0	0	0	34	0	0	0	0	0
0	0	0	0	0	34	0	0	0	0
0	0	0	0	0	0	34	0	0	0
0	0	0	0	0	0	0	34	0	0
0	0	0	0	0	0	0	0	34	0
0	0	0	0	0	0	0	0	0	0

Parameter a = -3000

Number of iterations to 2 significant digits = 3

Matrix Answer X =

$$\begin{bmatrix} 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 \end{bmatrix}$$

TABLE IV

Eq (4.110) to Solve Square Root of a Matrix

Example 1Matrix A =  $0_{10 \times 10}$ Matrix B =  $0_{10 \times 10}$ 

Matrix C =

10	.1	.2	.3	.4	.5	.6	.7	.8	.9
.1	20	.2	.3	.4	.5	.6	.7	.8	.9
.1	.2	30	.3	.4	.5	.6	.7	.8	.9
.1	.2	.3	40	.4	.5	.6	.7	.8	.9
.1	.2	.3	.4	50	.5	.6	.7	.8	.9
.1	.2	.3	.4	.5	60	.6	.7	.8	.9
.1	.2	.3	.4	.5	.6	70	.7	.8	.9
.1	.2	.3	.4	.5	.6	.7	80	.8	.9
.1	.2	.3	.4	.5	.6	.7	.8	90	.9
.1	.2	.3	.4	.5	.6	.7	.8	.9	100

Matrix D =  $-I_{10 \times 10}$ Matrix  $X_0 = 0_{10 \times 10}$ 

Parameter a = -30

Number of iterations to 3 significant digits = 17

Matrix Answer =

3.16	.012	.022	.031	.038	.045	.051	.057	.062	.067
.013	4.47	.019	.027	.034	.040	.046	.051	.056	.061
.011	.010	5.48	.025	.031	.037	.042	.048	.053	.057
.010	.018	.025	6.32	.029	.035	.040	.045	.050	.054
.009	.017	.023	.029	7.07	.033	.038	.043	.047	.052
.009	.016	.022	.028	.033	7.74	.037	.041	.046	.050
.008	.015	.021	.027	.032	.037	8.37	.040	.044	.048
.008	.015	.020	.026	.031	.035	.040	8.94	.043	.047
.008	.014	.020	.025	.030	.034	.039	.043	9.49	.046
.007	.013	.019	.024	.029	.033	.038	.041	.046	10.0



## V. CONCLUSION

The kinematic similarity theorems along with methods to compute a fundamental solution to the differential system of Eq (1.1) should prove beneficial for a wide variety of applications. Close scrutiny of the numerous examples should show the adaptability of these algorithms to a digital computer. In particular, the implementation of these theorems for the time dependent case should be notably more efficient than the time consuming iterative methods that yield questionably accurate solutions. For the constant cases of Eq (1.1), the application of these theorems could greatly simplify the original differential system so that conventional solution methods, which might not have otherwise worked, could be applied to the kinematically similar differential system. Finally, although further research is needed on the iterative methods used to solve the Liapunov and Riccati equations, these iterative schemes are able to find solutions that are not readily found by other methods.



### Bibliography

1. Loewy, H.A. "Uber Matrizen-und Differentialkomplexe" Math. Ann., 78: 1-51, 343-368 (1918).
2. Coppel, W.A. "Dichotomies and Reducibility". Journal of Differential Equations, 3: 500-521 (1967).
3. Berkey, D.D. "Block Diagonal Dominance and Reducibility for Linear Differential Systems." Journal of Mathematical Analysis and Applications, 56: 233-251 (1976).
4. Leuthauser, J.L. Solutions of Certain Matrix Equations with Application to Engineering Sciences. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology (December 1976).
5. Porter, B. "Design of Time-Optimal Regulators for Linear Multivariable Discrete-Time Plants." Electronics Letters, 12: 196-197 (April 1976).
6. Roth, W.E. "The Equations  $AX-YB=C$  and  $AX-XB=C$  in Matrices." Proceedings of the American Mathematical Society, 3: 392-396 (1952).
7. Nering, E.D. Linear Algebra and Matrix Theory. New York: John Wiley and Sons, 1970.
8. Bronson, R. Differential Equations. New York: McGraw-Hill Book Co., Inc., 1973.
9. McClelland, M.T. The Exact Solution of Systems of Linear Equations with Polynomial Coefficients. Ph.D. Thesis. The University of Wisconsin: Madison, Wisconsin (September 1971).
10. McClelland, M.T. A Comparison of Algorithms for the Exact Solution of Linear Equations. NTIS, PB-228-460 (January 1974).
11. Noble, B., Applied Linear Algebra. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1969.
12. Meirovitch, L. Methods of Analytical Dynamics. New York: McGraw-Hill, Inc., 1970.
13. LaSalle, J. and S. Lefschetz. Stability by Liapunov's Direct Method. New York: Academic Press, Inc., 1961.
14. Ma, E.C. "A Finite Series Solution of the Matrix Equation  $AX-XB=C$ ". SIAM Journal on Applied Mathematics, 14: 490-495 (May 1966).
15. Kitagawa, G. "An Algorithm for Solving the Matrix Equation  $X=FXF^T+S$ ". International Journal of Control, 25, #5, 745-753 (May 1977).

16. Lancaster, P. "Explicit Solutions of Linear Matrix Equations." SIAM Review, 12: 544-566 (October 1970).
17. Ben-Israel, B. and T.N. Greville. Generalized Inverses. New York: John Wiley and Sons, 1974.
18. Kalman, R.E. "A New Approach to Linear Filtering and Prediction Problems". ASME Journal of Basic Engineering, 82: 35-46 (March 1960).
19. Bryson, A.E., and Y.C. Ho. Applied Optimal Control. Washington D.C.: Hemisphere Publishing Corporation, 1975.
20. Breakwell, J.V. "The Optimization of Trajectories". SIAM Journal, 7: 215-247 (June 1959).
21. McReynolds, S.R. "The Successive Sweep Method and Dynamic Programming." Journal of Mathematical Analysis and Applications, 19: 565-598 (September 1967).
22. Jones, J., Jr., "Solution of Certain Matrix Equations." Proceedings of the American Mathematical Society, 31: 333-339 (1972).
23. Jones, J., Jr., J. Lukes, K. Dolan. "On Kinematic Similarity of Systems of Differential Equations." Notices of the American Mathematical Society, 24, #5: A-474 (August 1977).
24. Carnhahan, B. Applied Numerical Methods. New York: John Wiley and Sons, 1969.
25. Dolan, K. "Kinematic Similarities of Systems of Differential Equations." Presentation to the Fall 1977 Meeting of MAA, Ohio Section, Wright State University ( 29 October 1977 ).

## VITA

Captain Kevin Dolan was born in Boston, Mass. in 1947. After graduating from Catholic Memorial High School there, he attended USAFA from which he received a B. S. degree in Mathematics in 1969. Subsequent to pilot training at Moody AFB, Ga., Captain Dolan flew KC-135's at Travis AFB, Ca. before an assignment to Tan Son Nhut AB, S. Vietnam flying EC-47's. He returned to flying the KC-135 at Wurtsmith AFB, Mich. in 1973 and remained there until his assignment to AFIT in May 1976

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/GCS/MA/77D-1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ALGORITHMIC APPROACH FOR REDUCING KINEMATICALLY SIMILAR DIFFERENTIAL SYSTEMS		5. TYPE OF REPORT & PERIOD COVERED MS Thesis
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Kevin Dolan Captain		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology (AFIT-EN) Wright-Patterson AFB, Ohio 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Flight Control Division (AFFDL/FG) Air Force Flight Dynamics Laboratory Wright-Patterson AFB, Ohio 45433		12. REPORT DATE December 1977
		13. NUMBER OF PAGES 85
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES Approved for public release; IAW AFR 190-17 <i>J. F. Guess</i> JERRAL F. GUESS, Captain, USAF Director of Information		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Matrices (Mathematics)      Nonlinear Algebraic Equations Matrix Equations      Liapunov Equations      Nonlinear Differential Matrix Factorization      Riccati Equations      Equations Linear Equations      Quadratic Equations Linear Algebraic Equations      Linear Differential Equations		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Methods of reducing linear homogeneous differential matrix equations of the form $\dot{X} = A(t)X$ by kinematically similar transformations are presented in this thesis. Classical equivalence and similarity relationships between matrices with elements belonging to the polynomial domain with coefficients belonging to the field of real numbers are obtained which give necessary and sufficient conditions for the existence of solutions to certain polynomial equations. Block reduction techniques using kinematically similar transformations lead to a discussion of iterative and non-iterative solution methods for the Liapunov matrix equation, Liapunov		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)




UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

Block 20.

differential matrix equation, Riccati matrix equation, and the Riccati differential matrix equation.



UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)